Chapter 11.1 and 11.2

Exponents and Radicals

Ch. 11.1: Simplifying Expressions with Integral Exponents

- In this section, we review the laws of exponents and the concept of the negative exponent.

Laws of Exponents

- Product Law: $a^m \times a^n = a^{m+n}$
- Quotient Law: $\dfrac{a^m}{a^n} = a^{m-n}$ $\quad m > n,\ a \neq 0$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad m < n,\ a \neq 0$$

Give examples

Collect on board

Laws of Exponents (*continued*)

- Power Law:

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad b \neq 0$$

Zero & Negative Exponents

- A zero exponent is defined by:

$$a^0 = 1 \qquad a \neq 0$$

- A negative exponent is defined by:

$$a^{-n} = \frac{1}{a^n} \qquad a \neq 0$$

The Negative Exponent

- Negative exponents are generally not used in the expression of a final result, unless specified otherwise.
- When a factor is moved from the denominator to the numerator of a fraction, or conversely, the sign of the exponent is changed.

Example

- Express the given expression in simplest form with only positive exponents.

$$\frac{x^{-1}-y}{x-y^{-1}} \Rightarrow = \frac{\frac{1}{x}-y}{x-\frac{1}{y}} \Rightarrow = \frac{\frac{1-xy}{x}}{\frac{xy-1}{y}} \Rightarrow = \left(\frac{1-xy}{x}\right)\left(\frac{y}{xy-1}\right)$$

$$\Rightarrow = -\frac{y}{x}$$

Examples

? Ex 11.1 q 10,12,23,25

In Exercises 5–52, express each of the given expressions in simplest form with only positive exponents.

5. x^7x^{-4}
6. y^9y^{-2}
7. $2a^2a^{-6}$
8. $5ss^{-5}$
9. $5^0 \times 5^{-3}$
✓ **10.** $(3^2 \times 4^{-3})^3$
11. $(2\pi x^{-1})^2$
12. $(3xy^{-2})^3$
13. $2(5an^{-2})^{-1}$
14. $4(6s^2t^{-1})^{-2}$
✓ **15.** $(-4)^0$
✓ **16.** -4^0
17. $-7x^0$
18. $(-7x)^0$
19. $3x^{-2}$
20. $(3x)^{-2}$
21. $(7a^{-1}x)^{-3}$
✓ **22.** $7a^{-1}x^{-3}$
23. $\left(\frac{2}{n^3}\right)^{-3}$
24. $\left(\frac{3}{x^3}\right)^{-2}$
25. $3\left(\frac{a}{b^{-2}}\right)^{-3}$

A

10. $\left(3^2 \times 4^{-3}\right)^3 = \left(\frac{3^2}{4^3}\right)^3 = \frac{3^6}{4^9}$

11. $(2\pi x^{-1})^2 = \left(\frac{2\pi}{x}\right)^2 = \frac{4\pi^2}{x^2}$

12. $\left(3xy^{-2}\right)^3 = 3^3 x^3 y^{-6} = \frac{27x^3}{6}$

13. $2\left(5an^{-2}\right)^{-1} = 2 \times 5^{-1} a^{-1} n^{(-2)(-1)} = \frac{2n^2}{5a}$

14. $4\left(6s^2t^{-1}\right)^{-2} = 4\left(\frac{6s^2}{t}\right)^2 = \frac{4 \times 6^{-2} s^{-4}}{t^{-2}} = \frac{4t^2}{6^2 s^4} = \frac{t^2}{9s^4}$

15. $(-4)^0 = 1$

16. $-4^0 = -(1) = -1$

17. $-7x^0 = -7 \times 1 = -7$

18. $(-7x)^0 = 1$

19. $3x^{-2} = 3\left(\frac{1}{x^2}\right) = \frac{3}{x^2}$

20. $(3x)^{-2} = 3^{-2} x^{-2} = \frac{1}{9x^2}$

21. $\left(7a^{-1}x\right)^3 = 7^{-3} a^3 x^{-3} = \frac{a^3}{7^3 x^3} = \frac{a^3}{343x^3}$

22. $7a^{-1}x^{-3} = \frac{7}{ax^3}$

23. $\left(\frac{2}{n^3}\right)^{-3} = \frac{2^{-3}}{n^{-9}} = \frac{n^9}{8}$

24. $\left(\frac{3}{x^3}\right)^{-2} = \frac{3^{-2}}{x^{-6}} = \frac{x^6}{9}$

25. $\left(\frac{a}{b^{-2}}\right)^{-3} = \frac{3a^{-3}}{\left(b^{-2}\right)^{-3}} = \frac{\frac{3}{a^3}}{b^{(-2)(-3)}} = \frac{\frac{3}{a^3}}{b^6} = \frac{3}{a^3b^6}$

Ch. 11.2: Fractional Exponents

- Radical & Exponential Forms

$$a^{1/n} = \sqrt[n]{a}$$

Exponential form ($a^{1/n}$)

Radical form ($\sqrt[n]{a}$)

$$\sqrt{9} = \sqrt[2]{9} = 9^{\frac{1}{2}} = 3$$

$$\sqrt[4]{6} = 6^{\frac{1}{4}}$$

Examples

1. $\sqrt[3]{x} = x^{1/3}$

2. $\sqrt{x} = x^{1/2}$

Fractional Exponents

- If a fractional exponent is negative, it does not mean that the expression is negative.
- ***Example***:

$$y^{-1/2} = \frac{1}{y^{\frac{1}{2}}} = \frac{1}{\sqrt{y}}$$

Fractional Exponents

$$(a^m)^n = a^{mn}$$

- Applying the product law:

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Example 1

$$(a^m)^n = a^{mn}$$

- Simplify:

$$x^{2/3} = (x^{1/3})^2 = (\sqrt[3]{x})^2$$

or, $x^{2/3} = (x^2)^{1/3} = (\sqrt[3]{x})^2$

Example 2

$$(a^m)^n = a^{mn}$$

- Simplify:

$$27^{2/3} = (\sqrt[3]{27})^2$$

$$= (\sqrt[3]{3 \times 3 \times 3})^2$$

$$= 3^2$$

$$= 9$$

- Generally, it is easier to take the root first.

Examples

? Ex 11.2,q 7,13,21,28

In Exercises 5–28, evaluate the given expressions.

5. $25^{1/2}$

6. $27^{1/3}$

7. $81^{1/4}$

8. $125^{2/3}$

9. $100^{25/2}$

10. $-16^{5/4}$

11. $8^{-1/3}$

12. $16^{-1/4}$

13. $64^{-2/3}$

14. $-32^{-4/5}$

15. $5^{1/2}5^{3/2}$

16. $(4^4)^{3/2}$

17. $(3^6)^{2/3}$

18. $\dfrac{121^{-1/2}}{100^{1/2}}$

19. $\dfrac{1000^{1/3}}{-400^{-1/2}}$

20. $\dfrac{-7^{-1/2}}{6^{-1}7^{1/2}}$

21. $\dfrac{15^{2/3}}{5^2 15^{-1/3}}$

22. $\dfrac{(-27)^{1/3}}{6}$

23. $\dfrac{(-8)^{2/3}}{-2}$

24. $\dfrac{-4^{-1/2}}{(-64)^{-2/3}}$

25. $125^{-2/3} - 100^{-3/2}$

26. $32^{0.4} + 25^{-0.5}$

27. $\dfrac{16^{-0.25}}{5} + \dfrac{2^{-0.6}}{2^{0.4}}$

28. $\dfrac{4^{-1}}{36^{-1/2}} - \dfrac{5^{-1/2}}{5^{1/2}}$

A

7. $(81)^{1/4} = \sqrt[4]{81} = 3$

13. $64^{-2/3} = \frac{1}{(64^{1/3})^2} = \frac{1}{(\sqrt[3]{64})^2} = \frac{1}{4^2} = \frac{1}{16}$

21. $\frac{15^{2/3}}{5^2 \times 15^{-1/3}} = \frac{15^{2/3+1/3}}{5^2} = \frac{15^1}{25} = \frac{3}{5}$

28. $\frac{4^{-1}}{(36)^{-1/2}} - \frac{5^{-1/2}}{5^{1/2}} = \frac{(36)^{1/2}}{4} - \frac{1}{5^{1/2}5^{1/2}}$

$= \frac{\sqrt[2]{36}}{4} - \frac{1}{5}$

$= \frac{6}{4} - \frac{1}{5} = \frac{3}{2} - \frac{1}{5}$

$= \frac{3(5) - 1(2)}{10}$

$= \frac{15 - 2}{10} = \frac{13}{10}$

$$a^m \times a^n = a^{m+n}$$

$$a^0 = 1$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

- Radical & Exponential Forms

$$a^{1/n} = \sqrt[n]{a}$$

Exponential form — *Radical form*

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

$$\sqrt{9} = \sqrt[2]{9} = 9^{\frac{1}{2}} = 3$$

Worked examples

EXAMPLE 2 Simplifying basic expressions

(a) Applying Eqs. (2) and (5), we have

$$\frac{a^2b^3c^0}{ab^7} = \frac{a^{2-1}(1)}{b^{7-3}} = \frac{a}{b^4}$$

(b) Applying Eqs. (4) and (3), and then (6), we have

$$(x^{-2}y)^3 = (x^{-2})^3(y^3) = x^{-6}y^3 = \frac{y^3}{x^6}$$

EXAMPLE 3 Removing perfect square factors

To simplify $\sqrt{75}$, we know that $75 = (25)(3)$ and that $\sqrt{25} = 5$. As in Eq. (10), we write

$$\sqrt{75} = \sqrt{(25)(3)} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$$

perfect square

NOTE ▸ This illustrates one step that should always be carried out in simplifying radicals: *Always remove all perfect nth-power factors from the radicand of a radical of order n.* ■

EXAMPLE 5 Reducing order of radical

(a) $\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{3/6} = 2^{1/2} = \sqrt{2}$

Here, we started with a sixth root and ended with a square root, thereby reducing the order of the radical. Fractional exponents are often helpful for this.

(b) $\sqrt[8]{16} = \sqrt[8]{2^4} = 2^{4/8} = 2^{1/2} = \sqrt{2}$

(c) $\dfrac{\sqrt[4]{9}}{\sqrt{3}} = \dfrac{\sqrt[4]{3^2}}{\sqrt{3}} = \dfrac{3^{2/4}}{3^{1/2}} = 1$

(d) $\dfrac{\sqrt[6]{8}}{\sqrt{7}} = \dfrac{\sqrt[6]{2^3}}{\sqrt{7}} = \dfrac{2^{1/2}}{7^{1/2}} = \sqrt{\dfrac{2}{7}}$

(e) $\sqrt[9]{27x^6y^{12}} = \sqrt[9]{3^3x^6y^9y^3} = 3^{3/9}x^{6/9}y^{9/9}y^{3/9} = 3^{1/3}x^{2/3}yy^{1/3}$

$= y\sqrt[3]{3x^2y}$

EXAMPLE 6 Rationalizing denominator

To write $\sqrt{\frac{2}{5}}$ in an equivalent form in which the denominator is not included under the radical sign, we *create a perfect square in the denominator* by multiplying the numerator and the denominator under the radical by 5. This gives us $\sqrt{\frac{10}{25}}$, which may be written as $\frac{1}{5}\sqrt{10}$ or $\frac{\sqrt{10}}{5}$. These steps are written as follows:

$$\sqrt{\frac{2}{5}} = \sqrt{\frac{2 \times 5}{5 \times 5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{\sqrt{25}} = \frac{\sqrt{10}}{5}$$

perfect square

EXAMPLE 8 Rationalizing expression—application

The period T (in s) for one cycle of a simple pendulum is given by $T = 2\pi\sqrt{L/g}$, where L is the length of the pendulum and g is the acceleration due to gravity. Rationalize the denominator on the right side of this equation if $L = 3.0$ ft and $g = 32$ ft/s^2.

Substituting, and then rationalizing, we have

$$T = 2\pi\sqrt{\frac{3.0}{32}} = 2\pi\sqrt{\frac{3.0 \times 2}{32 \times 2}}$$

$$= 2\pi\sqrt{\frac{6.0}{64}} = \frac{2\pi}{8}\sqrt{6.0}$$

$$= \frac{\pi}{4}\sqrt{6.0} = 1.9 \text{ s}$$

EXAMPLE 1 Meaning of fractional exponent

We now verify that Eq. (1) holds for the above definitions:

$$a^{1/4}a^{1/4}a^{1/4}a^{1/4} = a^{(1/4)+(1/4)+(1/4)+(1/4)} = a^1$$

Now, $a^{1/4} = \sqrt[4]{a}$ by definition. Also, by definition, $\sqrt[4]{a}\,\sqrt[4]{a}\,\sqrt[4]{a}\,\sqrt[4]{a} = a$. Equation (1) is thereby verified for $n = 4$ in Eq. (7).

Equation (3) is verified by the following:

$$(a^{1/4})(a^{1/4})(a^{1/4})(a^{1/4}) = (a^{1/4})^4 = a^1 = \left(\sqrt[4]{a}\right)^4$$

EXAMPLE 2 Interpretation of fractional exponent

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4 \quad \text{or} \quad 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

EXAMPLE 4 Evaluating

(a) $(16)^{3/4} = (16^{1/4})^3 = 2^3 = 8$

(b) $4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{2}$ **(c)** $9^{3/2} = (9^{1/2})^3 = 3^3 = 27$

We note in (b) that Eq. (6) must also hold for negative rational exponents. In writing $4^{-1/2}$ as $1/4^{1/2}$, the *sign* of the exponent is changed. ■

EXAMPLE 7 Simplifying expressions

(a) $(8a^2b^4)^{1/3} = [(8^{1/3})(a^2)^{1/3}(b^4)^{1/3}]$ using Eq. (4)

$= 2a^{2/3}b^{4/3}$ using Eqs. (7) and (3)

(b) $a^{3/4}a^{4/5} = a^{3/4+4/5} = a^{31/20}$ using Eq. (1)

(c) $$\left(\frac{4^{-9/2}x^{2/3}}{2^{3/2}x^{-1/3}}\right)^{2/3} = \left(\frac{x^{2/3}x^{1/3}}{2^{3/2}4^{9/2}}\right)^{2/3}$$ using Eq. (6)

$$= \left(\frac{x^{2/3+1/3}}{2^{3/2}4^{9/2}}\right)^{2/3}$$ using Eq. (1)

$$= \frac{x^{(1)(2/3)}}{2^{(3/2)(2/3)}4^{(9/2)(2/3)}}$$ using Eq. (4)

$$= \frac{x^{2/3}}{(2)(4^3)} = \frac{x^{2/3}}{128}$$

(d) $$(4x^4)^{-1/2} - 3x^{-3} = \frac{1}{(4x^4)^{1/2}} - \frac{3}{x^3}$$ using Eq. (6)

$$= \frac{1}{2x^2} - \frac{3}{x^3} = \frac{x-6}{2x^3}$$ using Eq. (7); common denominator ■