Chapter 13

Exponential and Logarithmic Functions

Ch 13.1: The Exponential Function

- From Chapter 11 – any rational number can be used as an exponent.
- We now let the **exponent be a variable** so we can define the exponential function as:

$$\mathbf{y} = \boldsymbol{b}^{x} \qquad \boldsymbol{b} > \mathbf{0}, \quad \boldsymbol{b} \neq \mathbf{1} \quad \textbf{\textit{and x is real}}$$

base

exponent

The Exponential Function

Evaluate the function $y = -2(4^x)$ for the given values of x.

(a) If $x = 2$, $y = -2(4^2) = -2(16) = -32$.

(b) If $x = -2$, $y = -2(4^{-2}) = -2/16 = -1/8$.

(c) If $x = 3/2$, $y = -2(4^{3/2}) = -2(8) = -16$.

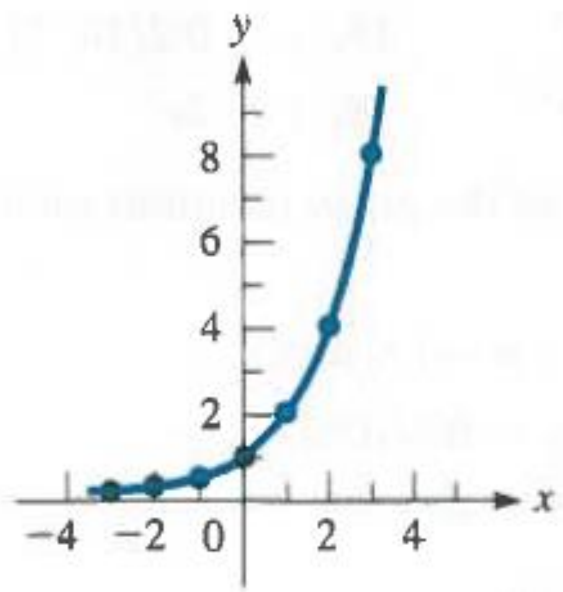
Example…

- Plot the graph of:

$$y = 2^x$$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$2^{-3} = \frac{1}{8}$ $\quad 2^0 = 1$ $\quad 2^3 = 8$



Examples to try…

In Exercises 7–12, evaluate the exponential function $y = 9^x$ for the given values of x.

7. $x = 0.5$ **8.** $x = 4$ **9.** $x = -2$

10. $x = -0.5$ **11.** $x = -3/2$ **12.** $x = 5/2$

In Exercises 13–18, plot the graphs of the given functions.

13. $y = 4^x$ **14.** $y = 0.25^x$ **15.** $y = 0.2(10^{-x})$

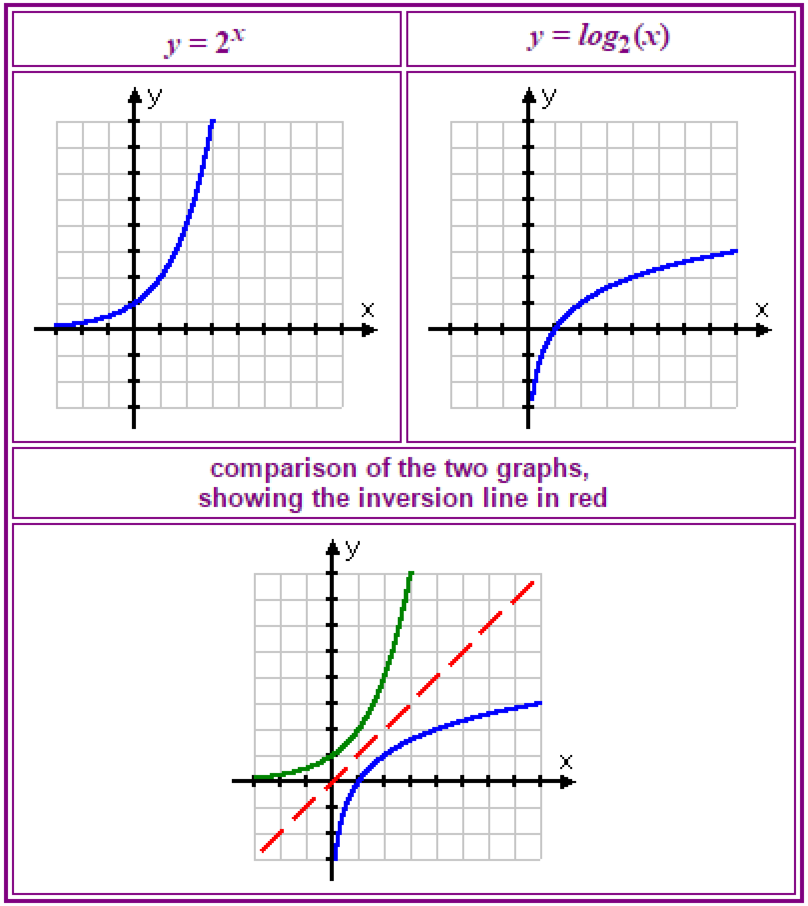
16. $y = -5(1.6^{-x})$ **17.** $y = 0.5\pi^x$ **18.** $y = 2e^x$

Ch. 13.2: Logarithmic Functions

- How do we solve equations where the exponent is the unknown?
- For example:
  - The formula for the growth of bacteria is $n = 1500(2)^t$ where n is the number of bacteria in t hours. How long will it take for 50 000 bacteria to grow?
  - That is: $50\ 000 = 1500(2)^t$
- We use logarithms to find the answer.

How do they look?

Logarithmic functions are the inverses of exponential functions



Logarithmic Functions

- Forms of a logarithm:

exponent

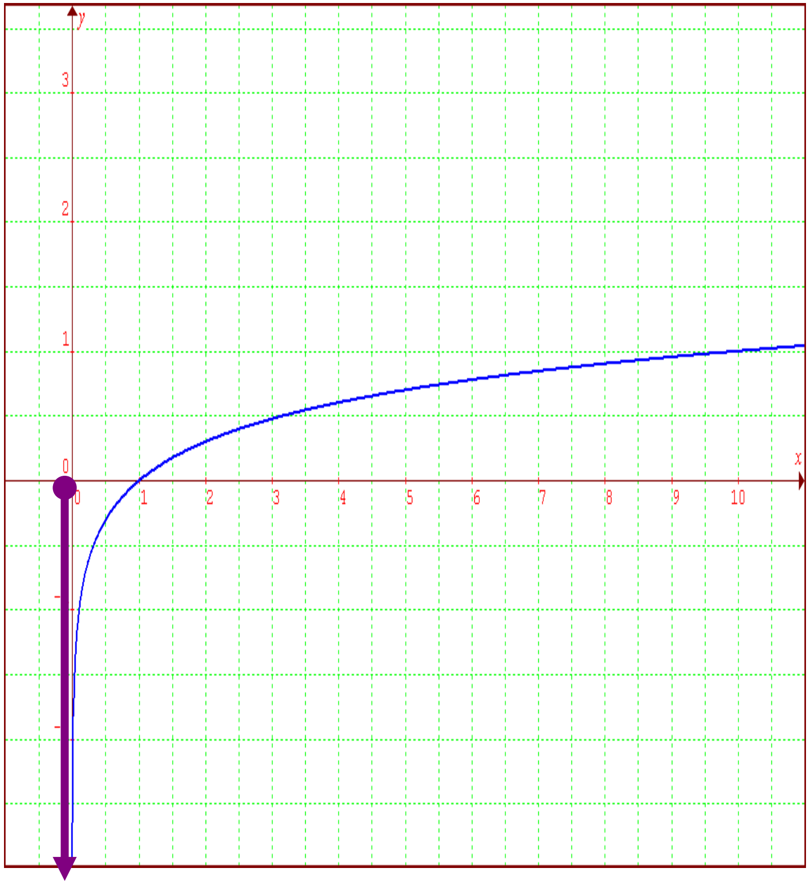
exponential form $\quad b^x = y \qquad\qquad x = \log_b y \quad$ logarithmic form

base

- Remember, exponents can be negative.

Graphing Logarithmic Functions

- Graphing: $y = \log_{10}x$
- Note the vertical ***asymptote*** along the negative y-axis where the graph ***never*** touches.



Basic Features of Logarithmic Functions (b > 1)

1. The domain is $x > 0$; the range is all values of y.
2. The negative y-axis is an asymptote of graph of $y = \log_b x$.
3. If $0 < x < 1$, $\log_b x < 0$; if $x = 1$, $\log_b x = 0$; if $x > 1$, $\log_b x > 0$.
4. If $x > 1$, x increases more rapidly than $\log_b x$.

? Ex 13.2 q 7, 14, 16, 25, 30

EXERCISES 13.2

In Exercises 1–4, perform the indicated operations if the given changes are made in the indicated examples of this section.

1. In Example 3(b), change the exponent to 4/5 and then make any other necessary changes.
2. In Example 4(b), change the 1/2 to 5/2 and then make any other necessary changes.
3. In Example 6, change the logarithm base to 4 and then make any other necessary changes.
4. In Example 7, change the logarithm base to 4 and then plot the graph.

In Exercises 5–16, express the given equations in logarithmic form.

5. $3^3 = 27$ **6.** $5^2 = 25$

7. $4^4 = 256$ **8.** $2^7 = 128$

9. $7^{-2} = \frac{1}{49}$ **10.** $3^{-2} = \frac{1}{9}$

11. $2^{-6} = \frac{1}{64}$ **12.** $(12)^0 = 1$

13. $8^{1/3} = 2$ **14.** $(81)^{3/4} = 27$

15. $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$ **16.** $\left(\frac{1}{2}\right)^{-2} = 4$

In Exercises 17–28, express the given equations in exponential form.

17. $\log_3 81 = 4$ **18.** $\log_{11} 121 = 2$

19. $\log_9 9 = 1$ **20.** $\log_{15} 1 = 0$

21. $\log_{25} 5 = \frac{1}{2}$ **22.** $\log_8 16 = \frac{4}{3}$

23. $\log_{243} 3 = 0.2$ **24.** $\log_{32}\left(\frac{1}{8}\right) = -0.6$

25. $\log_{10} 0.1 = -1$ **26.** $\log_7\left(\frac{1}{49}\right) = -2$

27. $\log_{0.5} 16 = -4$ **28.** $\log_{1/3} 3 = -1$

In Exercises 29–44, determine the value of the unknown.

29. $\log_4 16 = x$ **30.** $\log_5 125 = x$

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7. $4^1 = 256$ has base 4, exponent 4, and number 256.

$\log_4 256 = 4$

14. $(81)^{3/4} = 27$ has base 81, exponent $\frac{3}{4}$, and number 27.

$\log_{81} 27 = \frac{3}{4}$

16. $\left(\frac{1}{2}\right)^{-2} = 4$ has base $\frac{1}{2}$, exponent -2, and number 4.

$\log_{1/2} 4 = -2$

25. $\log_{10} 0.1 = -1$ has base 10, exponent -1, and number 0.1.

$0.1 = 10^{-1}$

30. $\log_5 125 = x$ has base 5, exponent x, and number 125.

$5^x = 125,\ x = 3$

Ch. 13.3: Properties of Logarithms

- Since a logarithm is an exponent, the properties of logarithms will be similar to those of exponents.
- We will compare the ***laws of exponents*** with the ***laws of logarithms***.

Logarithm of a Product

- Product Law of Exponents

$$b^u \times b^v$$

$$= b^{u+v}$$

Exponents are added

- The Logarithm of a Product

$$\log_b (u \times v)$$

$$= \log_b u + \log_b v$$

Logarithm of a Product

- **Example 1:**
  - Write as the sum or difference of 2 or more logarithms.

$$\log 5x = \log 5 + \log x$$

- **Example 2:**
  - Express as a single logarithm with a coefficient of 1.

$$\log 2 + \log 4 = \log (2 \times 4) = \log 8$$

Logarithm of a Quotient

- Quotient Law of Exponents

$$b^u \div b^v$$

$$= b^{u-v}$$

Exponents are subtracted

- Logarithm of a Quotient

$$\log_b (u \div v)$$

$$= \log_b u - \log_b v$$

Logarithm of a Quotient

Write as the sum or difference of 2 or more logarithms.

- **Example 3:** log (x/**5**) = log x - log **5**
- **Example 4:**

$$\log\left(\frac{11x}{6y}\right) = \log 11x - \log 6y$$

$$= (\log 11 + \log x) - (\log 6 + \log y)$$

Why the use of brackets?

Logarithm of a Quotient

- **Example 5:**

Express as a single logarithm with a coefficient of 1.

$\log 2 + \log 6 - \log 4$

$= \log (2 \times 6) - \log 4$

$= \log 12 - \log 4$

$$= \log \frac{12}{4} = \log 3$$

Logarithm of a Power

- Power Law of Exponents

$$(b^{u})^{n} = b^{u \times n}$$

- Logarithm of a Power

$$\log_b u^n$$

$$= n \log_b u$$

Logarithm of a Power

- **Example 6:**

  $\log 3^{12} = 12 \log 3$

- **Example 7:**

  $\log 3^{y} = y \log 3$

- **Example 8:**

Solve: $\mathbf{10 = \log u^{5}}$

Notice that we now have a product with the potential to divide and leave y by itself.

Properties of Logarithms

- **Example 9:**

Express as a single logarithm with a coefficient of 1.

$3\log x - 2\log y + 5\log z$

$= \log x^{3} - \log y^{2} + \log z^{5}$

$$= \log\left(\frac{x^3}{y^2}\right) + \log z^5 = \log\left(\frac{x^3 z^5}{y^2}\right)$$

Summary

- Remember the ***Order of Operations*** when working with the properties of logarithms.
- Avoid clearing your calculator screen after each calculation.

Examples

?Ex 13.3 q 14, 16, 25

In Exercises 9–20, express each as a sum, difference, or multiple of logarithms. See Example 2.

9. $\log_5 33$

10. $\log_3 14$

11. $\log_7\left(\frac{5}{3}\right)$

12. $\log_3\left(\frac{2}{11}\right)$

13. $\log_2(a^3)$

14. $2\log_8(n^5)$

15. $\log_6 abc$

16. $\log_2\left(\frac{xy}{z^2}\right)$

17. $8\log_5\sqrt[4]{y}$

18. $\log_4\sqrt[7]{x}$

19. $\log_2\left(\frac{\sqrt{x}}{a^2}\right)$

20. $\log_3\left(\frac{\sqrt[3]{y}}{7}\right)$

In Exercises 21–28, express each as the logarithm of a single quantity. See Example 3.

21. $\log_b a + \log_b c$

22. $\log_2 3 + \log_2 x$

23. $\log_5 9 - \log_5 3$

24. $-\log_8 R + \log_8 V$

25. $-\log_b\sqrt{x} + \log_b x^2$

26. $\log_4 3^3 + \log_4 9$

27. $2\log_e 2 + 3\log_e \pi$

28. $\frac{1}{2}\log_b a - 2\log_b 5$

A

14. $2\log_8\left(n^5\right) = 10\log_8 n$

16. $\log_2\left(\frac{xy}{x^2}\right) = \log_2 xy - \log_2 z^2$

$= \log_2 x + \log_2 y - 2\log_2 z$

25. $-\log_b \sqrt{x} + \log_b x^2 = \log_b \frac{x^2}{x^{1/2}} = \log_b x^{3/2}$

Useful equations for the rest of the chapter

exponent

exponential form $b^{x} = y$ $\qquad$ $x = \log_{b} y$ logarithmic form

base

$\log_b (u \times v)$

$= \log_b u + \log_b v$

$\log_b u^n$

$= n \log_b u$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$\log_b (u \div v)$

$= \log_b u - \log_b v$

$\log_b b = 1$

$\log 10 = 1$.

$\ln e = 1$.

Ch. 13.4: Logarithms to the Base 10

- A common logarithm has a base of **10**.

$$\log_{10} N = \log N$$

- If there is no base identified in the logarithmic form, then assume it is to the base **10**.

Common Logarithms

Calculator Use:

Use the log key on your calculator to solve for N.

Common Logarithms

- Example:
  - Find **log N** if **N = 260**.

- Solution:
  - Rounding to 3 decimal places, **log 260 = 2.415**.

Common Logarithms

- **Remember, when you are finding the logarithm of a number, you are finding the power to which 10 must be raised to give the answer.**
- For example, $\log 260 = 2.415$ means

$$10^{2.415} = 260$$

Finding Antilogarithms

- Given: **$\log N = 2.415$**
- What is **N**?
- Rearranging this into exponential form, **$10^{2.415} = N$**
- We use the **10^x** key on the calculator to find the answer.
- **N = 260** antilogarithm

Summary

- Common logarithms are logarithms to the base 10.
- They are readily calculated using a scientific calculator that has been preprogrammed for common logarithms.
- We use the properties of logarithms to solve equations using common logarithms

Examples

? Ex 13.4 q 9 , 15

In Exercises 3–12, find the common logarithm of each of the given numbers by using a calculator.

3. 567
4. 0.0640
5. 9.24×10^6
6. 3.19^3
7. 1.174^{-4}
8. 8.043×10^{-8}
9. $\cos 12.5°$
10. $\tan 12.6$
11. $\sqrt{274}$
12. $\log_2 16$

In Exercises 13–20, find the antilogarithm of each of the given logarithms by using a calculator.

13. 4.437
14. 0.929
15. −1.3045
16. −6.9788
17. 3.30112
18. 8.82436
19. −2.23746
20. −10.336

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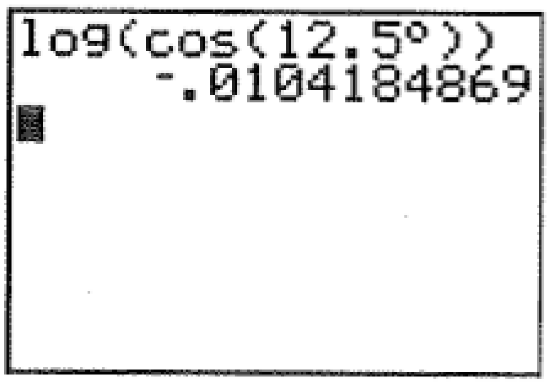
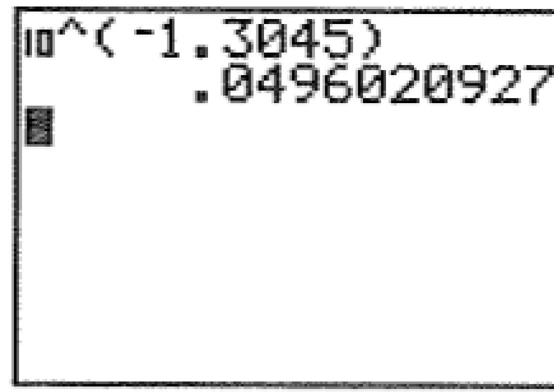
9. $\log\left(\cos 12.5^\circ\right) = -0.0104$



15. $10^{-1.3045} = 0.04960$



Ch. 13.5: Natural Logarithms

- A natural logarithm has a base of ***e***.

$$\log_e N = \mathit{ln}\ N$$

- Natural logarithms have widespread application in science and business.

Natural Logarithms

Calculator Use:

Use the *ln* key on your calculator to solve for N.

Useful equations for the rest of the chapter

exponent

exponential form $b^x = y$ $x = \log_b y$ logarithmic form

base

$\log_b (u \times v) = \log_b u + \log_b v$

$\log_b u^n = n \log_b u$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$\log_b (u \div v) = \log_b u - \log_b v$

$\log_b b = 1$
$\log 10 = 1$.
$\ln e = 1$.

Natural Logarithms

- Example:
  - Find $\ln$ **N** if **N** = 260.
- Answer:
  - To three decimal places: $\ln 260 = 5.561$

Natural Logarithms

- Remember, when you are finding the logarithm of a number, you are finding the power to which ***e*** must be raised to give the answer.
- For example, $\ln 260 = 5.561$ means

$$e^{5.560\,7} = 260$$

Natural Logarithms

- Example:
  - Find **N** if ***ln*** **N** is: 0.367

- Solution:
  - We take the antilogarithm of ***ln*** **N**
  - This gives us: $e^{0.367} = 1.443$
  - Therefore, $\ln 1.443 = 0.367$

Converting Logarithms

- Scientific calculators are programmed for logarithms in bases **10** & ***e***.
- We can ***solve*** any logarithm to a different base using the equation:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

a = 10

OR

a = e

Converting Logarithms

- Example: Find $\log_2 75$.
- Solution: We will find the answer in base **10**.

$$\log_b x = \frac{\log_a x}{\log_a b} \Rightarrow \log_2 75 = \frac{\log 75}{\log 2}$$

$$\Rightarrow \log_2 75 \cong 6.23$$

Summary

- Natural logarithms are logarithms to the base ***e***.
- They are readily calculated using a scientific calculator that has been preprogrammed for natural logarithms.
- We use the properties of logarithms to solve equations using natural logarithms

Examples

? Ex 13.5 q 19, 27

In Exercises 15–22, find the natural logarithms of the given numbers.

15. 51.4 **16.** 293 **17.** 1.394

18. 6552 **19.** 0.9917 **20.** 0.002086

21. $(0.012937)^4$ **22.** $\sqrt{0.000060808}$

In Exercises 27–34, find the natural antilogarithms of the given logarithms.

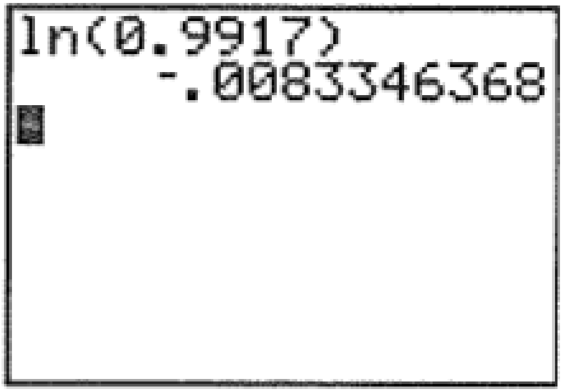
27. 2.190 **28.** 5.420 **29.** 0.0084210

30. 0.632 **31.** −0.7429 **32.** −2.94218

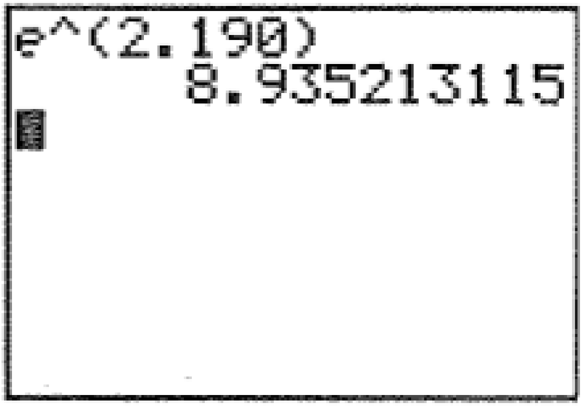
33. −23.504 **34.** −0.00804

A

19. $\ln 0.9917 = -0.008335$



27. $e^{2.190} = 8.935$



Ch. 13.6: Exponential and Logarithmic Equations

- In the fields of electronics and business, we are called upon to solve equations containing variable exponents or logarithms in some or all of the terms.

Solving Exponential & Logarithmic Equations

Our tools:

1. Converting between exponential form and logarithmic forms.
2. The Properties of Logarithms
3. The Identities in Logarithms.
4. Taking the logarithm of both sides.

exponent

exponential form $b^{x} = y$ $\qquad$ $x = \log_{b} y$ logarithmic form

base

$$\ln(e^{n}) = n \qquad\qquad e^{(\ln n)} = n$$

$$\log(10^{n}) = n \qquad\qquad 10^{(\log n)} = n$$

Radioactive decay

$$N = N_0 e^{\frac{-0.693\,t}{T_{1/2}}}$$

N = Number at time t
N_0 = Number at time t_0
$T_{1/2}$ = half life

If a source has a half life of 2000yrs. How many years will it take to decay to 10% of its original value?

$$\frac{1}{10} = e^{\frac{-0.693\,t}{T_{1/2}}}$$

$$\ln(0.1) = \frac{-0.693\,t}{2000}$$

t = 6,645 yrs

The Identities in Logarithms

- We use the identity: $\log_b b = 1$
- In common logarithms, this is $\log 10 = 1$.
- In natural logarithms, this is $\ln e = 1$.

$$b^x = y \quad then \quad x = log_b y$$

$$log_b(uv) = log_b(u) + log_b(v)$$

$$log_b\left(\frac{u}{v}\right) = log_b(u) - log_b(v)$$

$$log_b u^n = n log_b(u)$$

$$log_b x = \frac{log_a x}{log_a b}$$

$$log_e x = ln x$$

Exponential Equations

$$\log_b u^n = n\log_b(u)$$

- **Example 1:**
  - Solve for *x*. $2^{3x-1} = 6$
- **Solution:**
  - Take the logarithm of both sides.
    $\log(2^{3x-1}) = \log 6$
  - Apply the power law of logarithms.
    $(3x - 1)\log 2 = \log 6$
  - Solve. $x = 1.19$ (:to 2 decimal places)

Logarithmic Equations

1. We use algebra to isolate the logarithm with the unknown in it (x).
2. We convert the logarithmic equation into its exponential counterpart.

Logarithmic Equations

- **Example 2:**
  - $\log_x 64 = 3$
  - $x^3 = 64$
  - $\boldsymbol{x = 4}$
- **Example 3:**
  - $\log_{49} x = 1/2$
  - $49^{1/2} = x$
  - $\boldsymbol{x = 7}$

$$b^x = y \quad then \quad x = log_b y$$

$$log_b(uv) = log_b(u) + log_b(v)$$

$$log_b\left(\frac{u}{v}\right) = log_b(u) - log_b(v)$$

$$log_b u^n = n log_b(u)$$

$$log_b x = \frac{log_a x}{log_a b}$$

$$log_e x = ln x$$

Logarithmic Equations

$$b^x = y \quad then \quad x = log_b y$$

$$log_b(uv) = log_b(u) + log_b(v)$$

$$log_b\left(\frac{u}{v}\right) = log_b(u) - log_b(v)$$

$$log_b u^n = n log_b(u)$$

$$log_b x = \frac{log_a x}{log_a b}$$

$$log_e x = ln x$$

- **Example 4:**
  - $ln\ x - ln\ x^2 = ln\ 27$
  - $\mathbf{x = 1/27}$
- **Example 5:**
  - $\log(x^2 - 9) - 1 = \log(x + 3)$
  - $\mathbf{x = 13}$

$$\log(x^2-9) - 1 = \log(x+3)$$

note $\log(0)$ is undefined $x \neq -3$

$$\log(x^2-9) - \log(x+3) = 1$$

$$\log\left(\frac{x^2-9}{x+3}\right) = 1$$

$$10^{\left(\log\left(\frac{x^2-9}{x+3}\right)\right)} = 10^1$$

$$\frac{x^2-9}{x+3} = 10$$

$$x^2-9 = 10x+30$$

$$x^2-10x-39 = 0$$

$$(x-13)(x+3) = 0$$

$$x = 13$$

Summary

- Exponential & logarithmic equations are readily solved when we:

1. convert between exponential form and logarithmic forms.
2. apply the Properties of Logarithms
3. apply the Identities in Logarithms.
4. in some instances, take the logarithm of both sides.

Examples

? Ex 13.6 22, 29

$b^x = y \;\; then \;\; x = log_b y$

$log_b(uv) = log_b(u) + log_b(v)$

$log_b\left(\frac{u}{v}\right) = log_b(u) - log_b(v)$

$log_b u^n = n log_b(u)$

$log_b x = \frac{log_a x}{log_a b}$

$log_e x = ln x$

6. $\pi^x = 15$

7. $3^{-x} = 0.525$

9. $6^{x+1} = 78$

10. $5^{x-1} = 0.07$

12. $0.8^x = 0.4$

13. $0.6^x = 2^{x^2}$

15. $3 \log_8 x = -2$

16. $5 \log_{32} x = -3$

17. $\log x^2 = (\log x)^2$

18. $x^{\log x} = 1000x^2$

19. $\log_2 x + \log_2 7 = \log_2 21$

20. $2 \log_2 3 - \log_2 x = \log_2 45$

21. $2 \log(3 - x) = 1$

22. $3 \log(2x - 1) = 1$

23. $\log 12x^2 - \log 3x = 3$

24. $\ln x - \ln(1/3) = 1$

25. $3 \ln 2 + \ln(x - 1) = \ln 24$

26. $\log_2 x + \log_2(x + 2) = 3$

27. $\frac{1}{2} \log(x + 2) + \log 5 = 1$

28. $2 \log_x 2 + \log_2 x = 3$

29. $\log(2x - 1) + \log(x + 4) = 1$

30. $\ln(2x - 1) - 2 \ln 4 = 3 \ln 2$

29. $\log(2x-1)+\log(x+4)=1$

$$\log\left[(2x-1)(x+4)\right]=1$$

$$(2x-1)(x+4)=10$$

$$2x^2+7x-4=10$$

$$2x^2+7x-14=0$$

22. $3\log(2x-1)=1$

$$\log(2x-1)=\frac{1}{3}$$

$$2x-1=10^{1/3}=2.154$$

$$2x=2.154+1=3.154$$

$$x=\frac{3.154}{2}=1.58$$

Use the quadratic formula to solve for x:

$$x=\frac{-7\pm\sqrt{49-4(2)(-14)}}{2(2)}=\frac{-7\pm\sqrt{161}}{4}$$

$$=\frac{-7\pm12.689}{4}=-4.92,\ 142$$

$x=1.42$ (Since logs are not defined on negatives.)

A

exponent

exponential form $b^{x} = y$ $x = \log_{b} y$ logarithmic form

base

$\log_b (u \times v)$

$= \log_b u + \log_b v$

$\log_b u^n$

$= n \log_b u$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$\log_b (u \div v)$

$= \log_b u - \log_b v$

$\log_b b = 1$

$\log 10 = 1$.

$\ln e = 1$.

Exponential function	$y = b^x$
Logarithmic form	$x = \log_b y$
Logarithmic function	$y = \log_b x$
Laws of exponents	$b^u b^v = b^{u+v}$
	$\frac{b^u}{b^v} = b^{u-v}$
	$(b^u)^n = b^{nu}$
Properties of logarithms	$\log_b xy = \log_b x + \log_b y$
	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
	$\log_b(x^n) = n \log_b x$
	$\log_b 1 = 0 \qquad \log_b b = 1$
	$\log_b(b^n) = n$
Changing base of logarithms	$\log_b x = \frac{\log_a x}{\log_a b}$
	$\ln x = \frac{\log x}{\log e}$
	$\log x = \frac{\ln x}{\ln 10}$

Worked examples

EXAMPLE 1 Exponential functions

From the definition, $y = 3^x$ is an exponential function, but $y = (-3)^x$ is not because the base is negative. However, $y = -3^x$ is an exponential function, because it is -1 times 3^x, and any real-number multiple of an exponential function is also an exponential function.

Also, $y = \left(\sqrt{3}\right)^x$ is an exponential function since it can be written as $y = 3^{x/2}$. As long as x is a real number, so is $x/2$. Therefore, the exponent of 3 is real.

The function $y = 3^{-x}$ is an exponential function. If x is real, so is $-x$.

Other exponential functions are: $y = -2(8^{-0.55x})$ and $y = 35(1.0001)^x$.

EXAMPLE 2 Evaluating an exponential function

Evaluate the function $y = -2(4^x)$ for the given values of x.

(a) If $x = 2$, $y = -2(4^2) = -2(16) = -32$.

(b) If $x = -2$, $y = -2(4^{-2}) = -2/16 = -1/8$.

(c) If $x = 3/2$, $y = -2(4^{3/2}) = -2(8) = -16$.

(d) If $x = \sqrt{2}$, $y = -2(4^{\sqrt{2}}) = -14.206$ (calculator evaluation—see Fig. 1).

(e) If $x = \pi$, $y = -2(4^\pi) = -155.76$ (See Fig. 1).

EXAMPLE 3 Graphing an exponential function

Plot the graph of $y = 2^x$.

For this function, we have the values in the following table:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$2^{-3} = \frac{1}{8}$ $\quad 2^0 = 1$ $\quad 2^3 = 8$

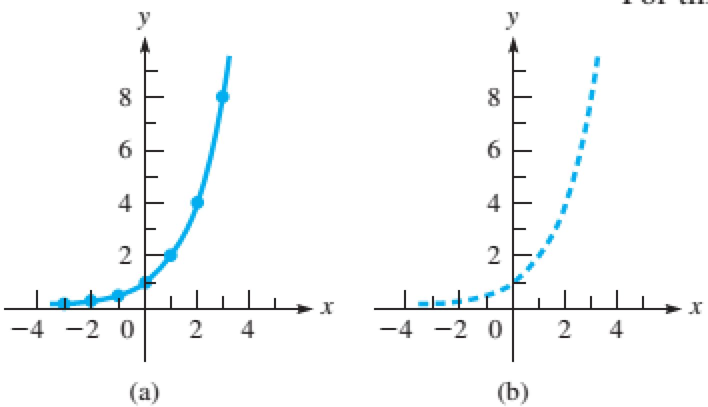


Fig. 2

The curve is shown in Fig. 2(a). Previously, we used only integral exponents, and the enlarged points are for these values. We previously introduced rational exponents, and using them would fill in many points between those for integers, but all the points for irrational numbers would be missing and the curve would be dotted (see Fig. 2(b)). Using all real numbers, including the irrational numbers, for exponents we have all points on the curve shown in Fig. 2(a).

We see that the x-axis is an *asymptote* of the curve. An asymptote is a line that the curve gets closer and closer to as values of x increase (or decrease) without bound, although the curve never actually touches the asymptote. ■

Practice Exercises

Evaluate $y = 16^x$ for:

1. $x = 3/2$ **2.** $x = -0.5$

EXAMPLE 1 Exponential form and logarithmic form

The equation $y = 2^x$ is written as $x = \log_2 y$ when written in logarithmic form. When we choose values of y to find corresponding values of x from this equation, we ask ourselves "2 raised to what power x gives y?"

This means that if $y = 8$, we ask "what power of 2 gives us 8?" Then knowing that $2^3 = 8$, we know that $x = 3$. Therefore, $3 = \log_2 8$. ■

EXAMPLE 3 Changing between forms

(a) $(64)^{1/3} = 4$ in logarithmic form is $\frac{1}{3} = \log_{64} 4$.

(b) $(32)^{3/5} = 8$ in logarithmic form is $\frac{3}{5} = \log_{32} 8$.

(c) $\log_2 32 = 5$ in exponential form is $32 = 2^5$.

(d) $\log_6 \left(\frac{1}{36}\right) = -2$ in exponential form is $\frac{1}{36} = 6^{-2}$.

(e) To change $4 \log_{16} 8 = 3$ to exponential form, first write it as $\log_{16} 8 = 3/4$. Then we can write the exponential form $8 = 16^{3/4}$.

EXAMPLE 4 Evaluating by changing form

(a) Find b, given that $-4 = \log_b\left(\frac{1}{81}\right)$.

Writing this in exponential form, we have $\frac{1}{81} = b^{-4}$. Thus, $\frac{1}{81} = \frac{1}{b^4}$ or $\frac{1}{3^4} = \frac{1}{b^4}$. Therefore, $b = 3$.

(b) Given $\log_4 y = 1/2$, in exponential form it becomes $y = 4^{1/2}$, or $y = 2$. ■

EXAMPLE 6 Logarithmic function

For the logarithmic function $y = \log_2 x$, we have the standard independent variable x and the standard dependent variable y.

If $x = 16$, $y = \log_2 16$, which means that $y = 4$, because $2^4 = 16$.

If $x = \frac{1}{16}$, $y = \log_2\left(\frac{1}{16}\right)$, which means that $y = -4$, because $2^{-4} = \frac{1}{16}$. ■

EXAMPLE 7 Graphing logarithmic function

Plot the graph of $y = \log_2 x$.

We can find the points for this graph more easily if we first put the equation in exponential form: $x = 2^y$. By assuming values for y, we can find the corresponding values for x.

$2^{-2} = \frac{1}{4}$ $\qquad$ $2^2 = 4$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-2	-1	0	1	2	3

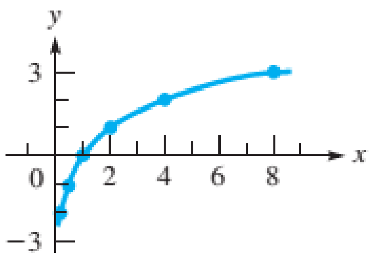


Fig. 8

Using these values, we construct the graph seen in Fig. 8.

EXAMPLE 9 Inverse functions

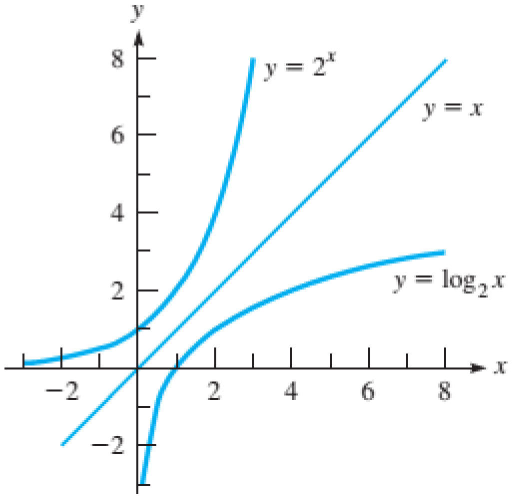


Fig. 10

The functions $y = 2^x$ and $y = \log_2 x$ are inverse functions. We show this by solving $y = 2^x$ for x and then interchanging x and y.

Writing $y = 2^x$ in logarithmic form gives us $x = \log_2 y$. Then interchanging x and y, we have $y = \log_2 x$, which is the inverse function.

Making a table of values for each function, we have

$y = 2^x$:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$y = \log_2 x$:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3

We see that the coordinates are interchanged. In Fig. 10, note that the graphs of these two functions reflect each other across the line $y = x$.

EXAMPLE 1 Sum of logarithms for product

We know that $8 \times 16 = 128$. Writing these numbers as powers of 2, we have

$$8 = 2^3 \qquad 16 = 2^4 \qquad 128 = 2^7 = 2^{3+4}$$

The logarithmic forms can be written as

$$3 = \log_2 8 \qquad 4 = \log_2 16 \qquad 3 + 4 = \log_2 128$$

This means that

$$\log_2 8 + \log_2 16 = \log_2 128$$

where

$$8 \times 16 = 128$$

The *sum of the logarithms* of 8 and 16 equals the logarithm of 128, where the *product* of 8 and 16 equals 128.

EXAMPLE 2 Logarithms of product, quotient, power

(a) Using Eq. (7), we may express $\log_4 15$ as a sum of logarithms:

$$\log_4 15 = \log_4(3 \times 5) = \log_4 3 + \log_4 5$$

logarithm of product
sum of logarithms

(b) Using Eq. (8), we may express $\log_4\left(\frac{5}{3}\right)$ as the difference of logarithms:

$$\log_4\left(\frac{5}{3}\right) = \log_4 5 - \log_4 3$$

logarithm of quotient
difference of logarithms

(c) Using Eq. (9), we may express $\log_4(t^2)$ as twice $\log_4 t$:

$$\log_4(t^2) = 2\log_4 t$$

logarithm of power
multiple of logarithm

(d) Using Eq. (8) and then Eq. (7), we have

$$\log_4\left(\frac{xy}{z}\right) = \log_4(xy) - \log_4 z$$
$$= \log_4 x + \log_4 y - \log_4 z$$

EXAMPLE 3 Sum of logarithms as single quantity

We may also express a sum or difference of logarithms as the logarithm of a single quantity.

(a) $\log_4 3 + \log_4 x = \log_4(3 \times x) = \log_4 3x$ using Eq. (7)

(b) $\log_4 3 - \log_4 x = \log_4\left(\dfrac{3}{x}\right)$ using Eq. (8)

(c) $\log_4 3 + 2\log_4 x = \log_4 3 + \log_4(x^2) = \log_4 3x^2$ using Eqs. (7) and (9)

(d) $\log_4 3 + 2\log_4 x - \log_4 y = \log_4\left(\dfrac{3x^2}{y}\right)$ using Eqs. (7), (8), and (9) ■

EXAMPLE 5 Using the properties of logarithms

(a) $\log_2 6 = \log_2(2 \times 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$

(b) $\log_5 \frac{1}{5} = \log_5 1 - \log_5 5 = 0 - 1 = -1$

(c) $\log_7 \sqrt{7} = \log_7(7^{1/2}) = \frac{1}{2}\log_7 7 = \frac{1}{2}$

EXAMPLE 7 Solving equation with logarithms

Use the basic properties of logarithms to solve the following equation for y in terms of x: $\log_b y = 2\log_b x + \log_b a$.

Using Eq. (9) and then Eq. (7), we have

$$\log_b y = \log_b(x^2) + \log_b a = \log_b(ax^2)$$

Because we have the logarithm to the base b of different expressions on each side of the resulting equation, the expressions must be equal. Therefore,

$$y = ax^2$$

EXAMPLE 1 Base 10 logarithms on calculator

Using a calculator to find log 426, as shown in the first two lines of the display in Fig. 11, we find that

$$\log 426 = 2.629$$

no base shown means base is 10

when the result is rounded off. The decimal part of a logarithm is normally expressed to the same accuracy as that of the number of which it is the logarithm, although showing one additional digit in the logarithm is generally acceptable.

Because $10^2 = 100$ and $10^3 = 1000$, and in this case

$$10^{2.629} = 426$$

we see that the 2.629 power of 10 gives a number between 100 and 1000.

EXAMPLE 2 Negative base 10 logarithm

Finding log 0.03654, as shown in the third and fourth lines of the calculator display in Fig. 11, we see that

$$\log 0.03654 = -1.4372$$

We note that the logarithm here is negative. This should be the case when we recall the meaning of a logarithm. Raising 10 to a negative power gives us a number between 0 and 1, and here we have

$$10^{-1.4372} = 0.03654$$

EXAMPLE 3 Antilogarithm—inverse logarithm

Given $\log N = 1.1854$, we find N as shown in the first two lines of the calculator display in Fig. 12. Therefore,

$$N = 15.32$$

where the result has been rounded off. Since $10^1 = 10$ and $10^2 = 100$, we see that $N = 10^{1.1854}$ and is a number between 10 and 100.

EXAMPLE 5 Calculation using logarithms

A certain computer design has 64 different sequences of ten binary digits so that the total number of possible states is $(2^{10})^{64} = 1024^{64}$. Evaluate 1024^{64} using logarithms. (It is very possible that your calculator cannot do this calculation directly.)

Because $\log x^n = n \log x$, we know that $\log 1024^{64} = 64 \log 1024$. Although most calculators will not directly evaluate 1024^{64}, we can use one to find the value of $64 \log 1024$. Because 1024^{64} is *exact*, we will show ten calculator digits until we round off the result. We therefore evaluate 1024^{64} as follows:

$$\begin{aligned}
\text{Let } N &= 1024^{64} \\
\log N &= \log 1024^{64} = 64 \log 1024 && \text{using Eq. (9): } \log_b x^n = n \log_b x \\
&= 192.6591972 \\
N &= 10^{192.6591972} && \text{meaning of logarithm} \\
&= 10^{192} \times 10^{0.6591972} && \text{using Eq. (4): } b^u b^v = b^{u+v} \\
&= (10^{192}) \times (4.5624) && \text{antilogarithm of 0.6591972 is 4.5624 (rounded off)} \\
&= 4.5624 \times 10^{192}
\end{aligned}$$

By using Eq. (4), $10^{0.6591972}$ represents a number between 1 and 10 ($10^0 = 1$ and $10^1 = 10$), and we can write the result immediately in scientific notation.

Although we used a calculator to find 192.6591972 and 4.5624 as shown in Fig. 13, the calculation was done essentially by logarithms. ■

EXAMPLE 1 Change of base to find natural log

Change log 20 to a logarithm with base e; that is, find ln 20.

Using Eq. (12) with $a = 10$, $b = e$, and $x = 20$, we have

$$\log_e 20 = \frac{\log_{10} 20}{\log_{10} e}$$

or

$$\ln 20 = \frac{\log 20}{\log e} = 2.996$$ see lines 1 and 2 of calculator display in Fig. 14

This means that $e^{2.996} = 20$. ■

EXAMPLE 2 Change of base to find log to base 5

Find $\log_5 560$.

In Eq. (12), if we let $a = 10$ and $b = 5$, we have

$$\log_5 x = \frac{\log x}{\log 5}$$

In this example, $x = 560$. Therefore, we have

$$\log_5 560 = \frac{\log 560}{\log 5} = 3.932$$ see lines 3 and 4 of calculator display in Fig. 14

From the definition of a logarithm, this means that

$$5^{3.932} = 560$$ ■

EXAMPLE 3 Natural log on calculator

(a) From the first two lines of the calculator display in Fig. 15, we find that

$$\ln 236.5 = 5.4659$$

which means that $e^{5.4659} = 236.5$.

(b) Given that $\ln N = -0.8729$, we determine N by finding $e^{-0.8729}$ on the calculator. This gives us (see lines 3 and 4 of the display in Fig. 15)

$$N = 0.4177$$

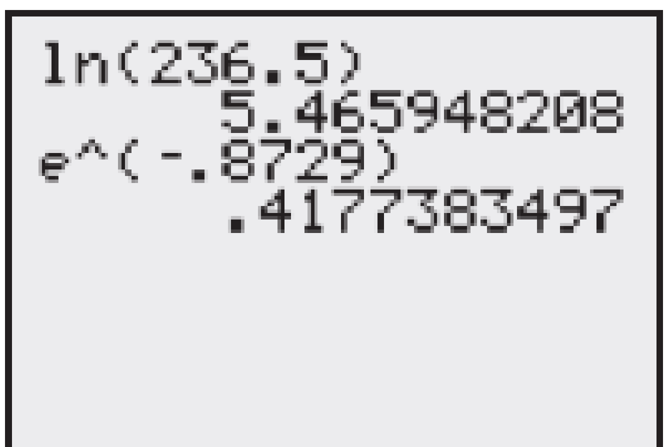


Fig. 15

EXAMPLE 2 Solving an exponential equation

Solve the equation $3^{x-2} = 5$.

Taking logarithms of each side and equating them, we have

$$\log 3^{x-2} = \log 5$$

$$(x-2)\log 3 = \log 5 \qquad \text{using Eq. (13.9)}$$

$$x = 2 + \frac{\log 5}{\log 3} = 3.465$$

This solution means that

$$3^{3.465-2} = 3^{1.465} = 5$$

which can be checked by a calculator.

$$log_b u^n = n log_b(u)$$

EXAMPLE 3 Solving an exponential equation

Solve the equation $2(4^{x-1}) = 17^x$.

By taking logarithms of each side, we have the following:

$$\log 2 + (x-1)\log 4 = x\log 17 \qquad \text{using Eqs. (13.7) and (13.9)}$$

$$x\log 4 - x\log 17 = \log 4 - \log 2$$

$$x(\log 4 - \log 17) = \log 4 - \log 2$$

$$x = \frac{\log 4 - \log 2}{\log 4 - \log 17} = \frac{\log(4/2)}{\log 4 - \log 17} \qquad \text{using Eq. (13.8)}$$

$$= \frac{\log 2}{\log 4 - \log 17} = -0.479$$

$$b^x = y \quad then \quad x = log_b y$$

$$log_b(uv) = log_b(u) + log_b(v)$$

$$log_b\left(\frac{u}{v}\right) = log_b(u) - log_b(v)$$

$$log_b u^n = n\,log_b(u)$$

$$log_b x = \frac{log_a x}{log_a b}$$

$$log_e x = ln x$$

EXAMPLE 1 Exponential equation—solved two ways

(a) We can solve the exponential equation $2^x = 8$ by writing it in logarithmic form. This gives

$$x = \log_2 8 = 3 \qquad 2^3 = 8$$

This method is good if we can directly evaluate the resulting logarithm.

(b) Because 2^x and 8 are equal, the logarithms of 2^x and 8 are also equal. Therefore, we can also solve $2^x = 8$ in a more general way by taking logarithms (to any proper base) of both sides and equating these logarithms. This gives us

$$\log 2^x = \log 8 \qquad \text{or} \qquad \ln 2^x = \ln 8$$

$$x \log 2 = \log 8 \qquad\qquad x \ln 2 = \ln 8 \qquad \text{using Eq. (9)}$$

$$x = \frac{\log 8}{\log 2} = 3 \qquad\qquad x = \frac{\ln 8}{\ln 2} = 3 \qquad \text{using a calculator}$$

EXAMPLE 3 Solving an exponential equation

Solve the equation $2(4^{x-1}) = 17^x$.

By taking logarithms of each side, we have the following:

$$\begin{aligned}
\log 2 + (x-1)\log 4 &= x \log 17 && \text{using Eqs. (7) and (9)} \\
x \log 4 - x \log 17 &= \log 4 - \log 2 \\
x(\log 4 - \log 17) &= \log 4 - \log 2 \\
x = \frac{\log 4 - \log 2}{\log 4 - \log 17} &= \frac{\log(4/2)}{\log 4 - \log 17} && \text{using Eq. (8)} \\
= \frac{\log 2}{\log 4 - \log 17} &= -0.479
\end{aligned}$$

EXAMPLE 4 Exponential equation—application

At constant temperature, the atmospheric pressure p (in Pa) at an altitude h (in m) is given by $p = p_0e^{kh}$, where p_0 is the pressure where $h = 0$ (usually taken as sea level). Given that $p_0 = 101.3$ kPa (atmospheric pressure at sea level) and $p = 68.9$ kPa for $h = 3050$ m, find the value of k.

Because the equation is defined in terms of e, we can solve it most easily by taking natural logarithms of *each side*. By doing this, we have the following solution:

$$\ln p = \ln(p_0e^{kh}) = \ln p_0 + \ln e^{kh} \qquad \text{using Eq. (7)}$$

$$= \ln p_0 + kh \ln e = \ln p_0 + kh \qquad \text{using Eq. (9), } \ln e = 1$$

$$\ln p - \ln p_0 = kh$$

$$k = \frac{\ln p - \ln p_0}{h}$$

Substituting the given values, we have

$$k = \frac{\ln(68.9 \times 10^3) - \ln(101.3 \times 10^3)}{3050} = -0.000126/\text{m}$$

EXAMPLE 7 Solving logarithmic equation

Solve the logarithmic equation $2 \ln 2 + \ln x = \ln 3$.

Using the properties of logarithms, we have the following solution:

$$
\begin{aligned}
2 \ln 2 + \ln x &= \ln 3 \\
\ln 2^2 + \ln x - \ln 3 &= 0 && \text{using Eq. (9)} \\
\ln \frac{4x}{3} &= 0 && \text{using Eqs. (7) and (8)} \\
\frac{4x}{3} &= e^0 = 1 && \text{exponential form} \\
4x &= 3, \qquad x = 3/4
\end{aligned}
$$

Because $\ln(3/4) = \ln 3 - \ln 4$, this solution checks in the *original equation*.

The calculator solution of this equation, using the *Solver* feature, is displayed in Fig. 18. It can also be solved graphically by finding the *zero* of the function $y_1 = 2 \ln 2 + \ln x - \ln 3$, or the *intersection* of $y_1 = 2 \ln 2 + \ln x$ and $y_2 = \ln 3$. ■

EXAMPLE 8 **Solving logarithmic equation**

Solve the logarithmic equation $2 \log x - 1 = \log(1 - 2x)$.

$$\log x^2 - \log(1 - 2x) = 1$$

$$\log \frac{x^2}{1 - 2x} = 1 \qquad \text{using Eq. (8)}$$

$$\frac{x^2}{1 - 2x} = 10^1 \qquad \text{exponential form}$$

$$x^2 = 10 - 20x$$

$$x^2 + 20x - 10 = 0$$

$$x = \frac{-20 \pm \sqrt{400 + 40}}{2} = -10 \pm \sqrt{110}$$

Because logarithms of negative numbers are not defined and $-10 - \sqrt{110}$ is negative and cannot be used in the first term of the original equation, we have

$$x = -10 + \sqrt{110} = 0.488 \qquad \text{use a calculator to check this result}$$