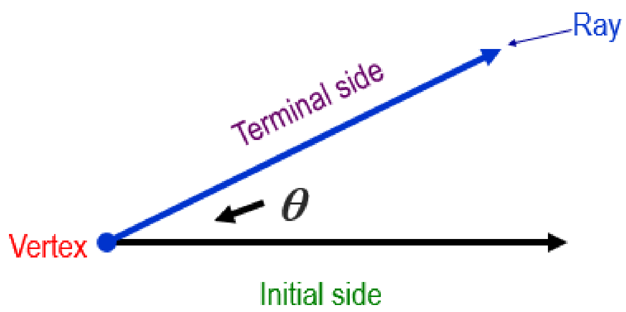
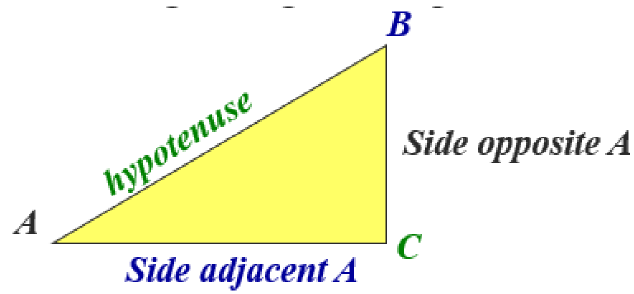
Chapter 4, 8 and 10

The Trigonometric Functions

Definitions

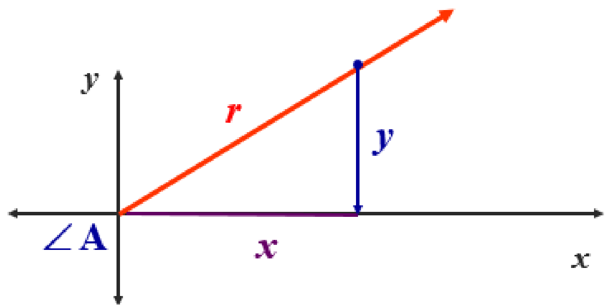




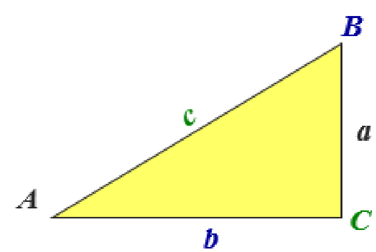
Coterminal
Positive = θ + 360
Negative = θ - 360

Pythagoras

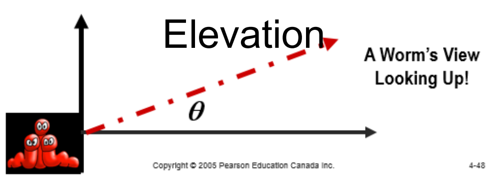
SOH CAH TOA
S=O/H, C=A/H, T=O/A



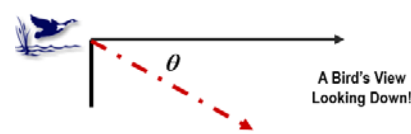
360° is 2π rads



- Sine of θ: $\sin\theta = \frac{y}{r}$
- Cosecant of θ: $\csc\theta = \frac{r}{y}$



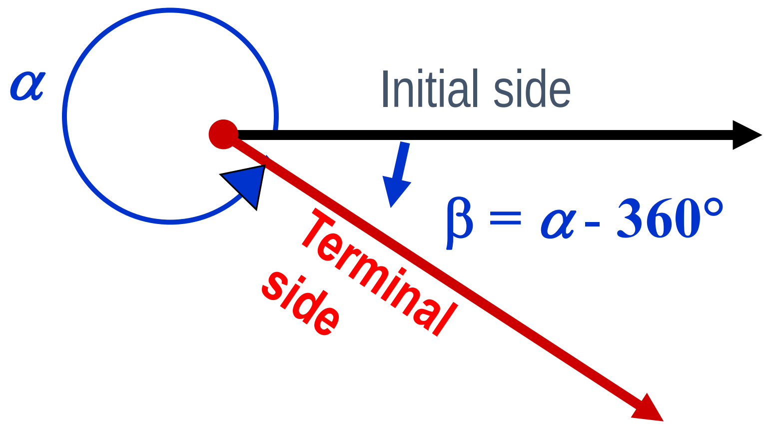
csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A



Depression

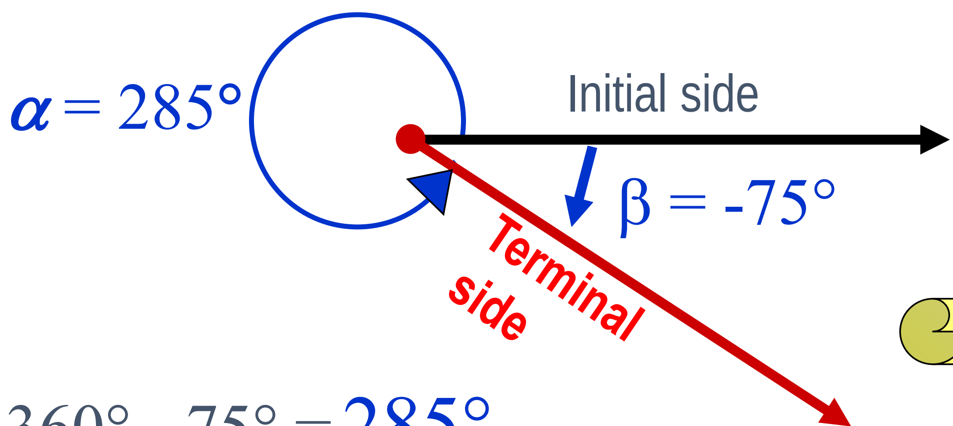
Angles Formed by Rotation

- Positive angles can also be written as ***negative*** angles: clockwise rotation
- Also, denoted by Greek letters.



Angles Formed by Rotation

- Write -75° as a positive angle.



$360° - 75° = 285°$

The angles, α and β, are called coterminal angles since they share the same initial & terminal sides.

Angle Conversions

- $360°$ = 2π radians = 1 revolution
- 1 degree = 60 minutes
- 1 minute = 60 seconds

$$65°25' = 65° + \frac{25}{60} = 65.42°$$

$$\begin{aligned} &32.459° \\ &= 32° + 0.459 \times 60 \\ &= 32° + 26.34' \end{aligned}$$

Convert 32.459° to degree/minute/second form.

$$
\begin{aligned}
&32.459^\circ \\
&= 32^\circ + 0.459 \times 60 \\
&= 32^\circ + 26.34' \\
&= 32^\circ 26' + 0.34 \times 60 \\
&= 32^\circ 26' 20.4''
\end{aligned}
$$

Convert to degree/minute/second form. Do examples then class do some

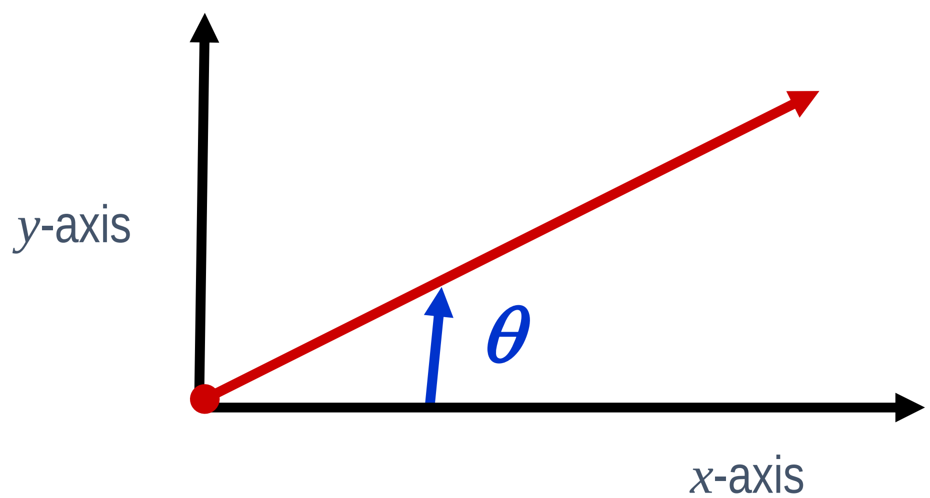
Angle Conversions

- There are **$360°$ in 2π rads** in a full circle.

Examples

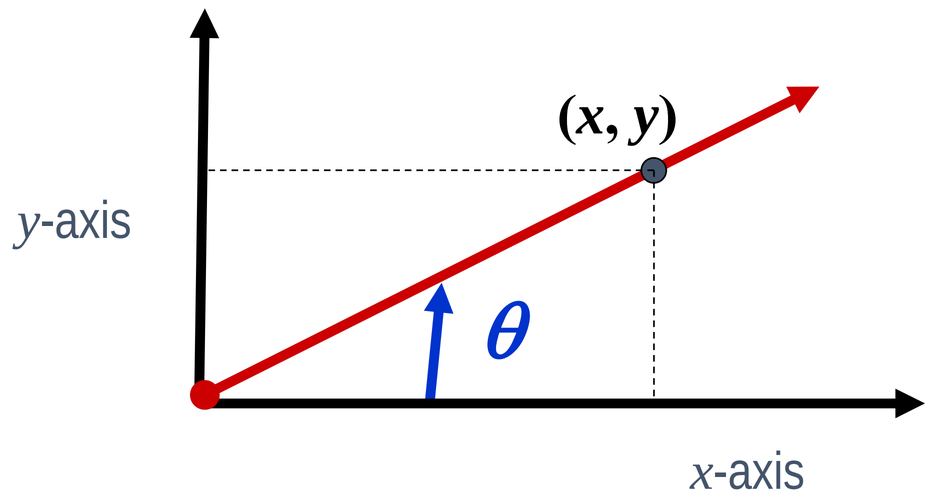
Standard Position of an Angle

- If the initial side of the angle is the positive ***x***-axis, and the vertex is the origin, the angle is said to be in ***standard position***.



Standard Position of an Angle

- The terminal side of an angle is uniquely determined by knowing that it passes through the point **(*x*, *y*)**.



Coterminal = same initial and same terminal

9. positive: $45^\circ + 360^\circ = 405^\circ$
negative: $45^\circ - 360^\circ = -315^\circ$

Note: can go round more than once

Ex 4.1 q 9-15 & 17-20

In Exercises 9–16, determine one positive and one negative coterminal angle for each angle given.

9. 45° **10.** 73°
11. −150° **12.** 462°
13. 70°30′ **14.** 153°47′
15. 278.1° **16.** −197.6°

In Exercises 17–20, by means of the definition of a radian, change the given angles in radians to equal angles expressed in degrees to the nearest 0.01°.

17. 0.265 rad **18.** 0.838 rad
19. 1.447 rad **20.** −3.642 rad

17. To change 0.265 rad to degrees multiply by $\frac{180}{\pi}$,

$$0.265 \text{ rad}\left(\frac{180^\circ}{\pi \text{ rad}}\right) \approx 15.18^\circ$$

9. positive: $45^\circ + 360^\circ = 405^\circ$

negative: $45^\circ - 360^\circ = -315^\circ$

A

9. positive: $45^\circ + 360^\circ = 405^\circ$

negative: $45^\circ - 360^\circ = -315^\circ$

10. $73^\circ + 360^\circ = 433^\circ$

$73^\circ - 360^\circ = -287^\circ$

11. $-150^\circ + 360^\circ = 210^\circ$

$-150^\circ - 360^\circ = -510^\circ$

12. $462^\circ + 360^\circ = 822^\circ$

$462^\circ - 2(360^\circ) = -258^\circ$

13. positive: $70^\circ 30' + 360 = 430^\circ 30'$

negative: $70^\circ 30' - 360^\circ = -289^\circ 30'$

14. $153^\circ 47' + 360^\circ = 513^\circ 47'$

$153^\circ 47' = -360^\circ = -206^\circ 13'$

15. $278.1^\circ + 360^\circ = 638.1^\circ$

$278.1^\circ - 360^\circ = -81.9^\circ$

17. To change 0.265 rad to degrees multiply by $\frac{180}{\pi}$,

$0.265 \text{ rad} \left(\frac{180^\circ}{\pi \text{ rad}}\right) \approx 15.18^\circ$

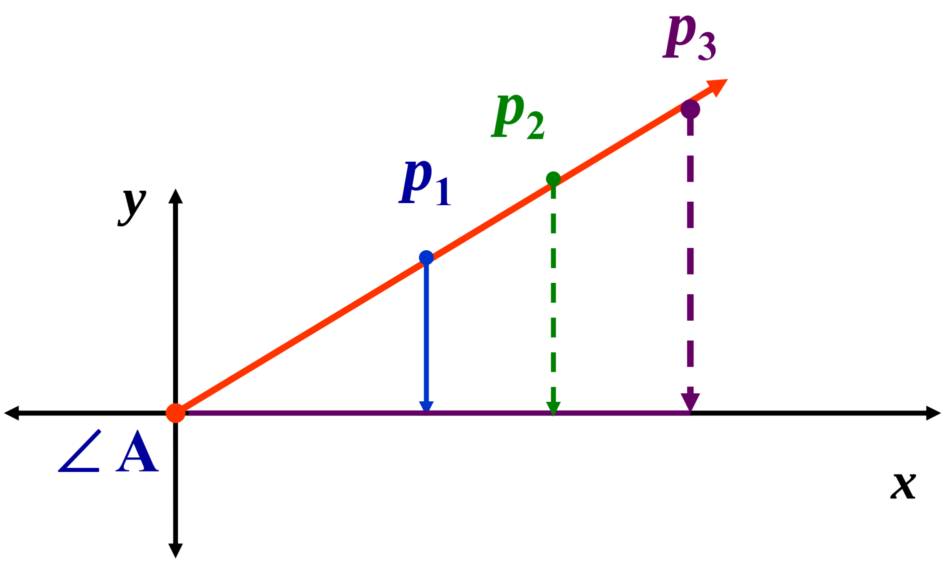
18. To change 0.838 rad to degrees multiply by $180^\circ / \pi$, which gives 48.01; $0.0838 \text{ rad} = 82.91^\circ$

19. To change 1.447 rad to degrees multiply by $180^\circ / \pi$, which gives 82.91; $1.447 \text{ rad} = 82.91^\circ$

20. To change -3.642 rad to degrees multiply by $180^\circ / \pi$, which gives -208.67; $-3.642 \text{ rad} = -208.67^\circ$

Determining the Trigonometric Ratios

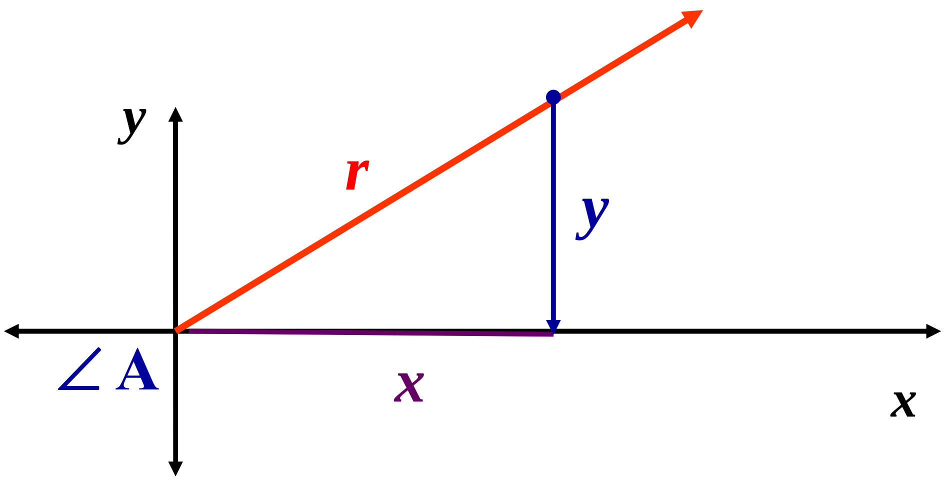
Regardless how far ***p*** is away from ∠ A, the size of the angle will never change.



Therefore, the ***ratio*** between the lengths of the sides will never change.

Determining the Trigonometric Ratios

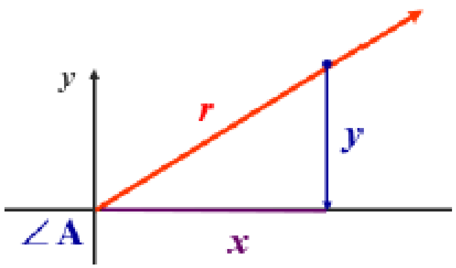
We label the right triangle as:



The trigonometric ratios are defined as follows:

The Trigonometric Functions

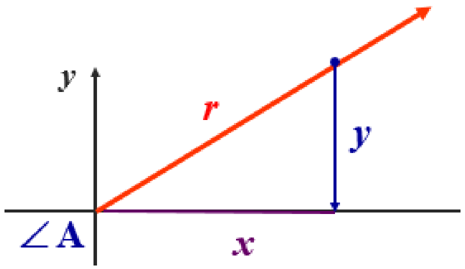
- Sine of θ: $\sin\theta = \dfrac{y}{r}$
- Cosecant of θ: $\csc\theta = \dfrac{r}{y}$
- Cosine of θ: $\cos\theta = \dfrac{x}{r}$
- Secant of θ: $\sec\theta = \dfrac{r}{x}$
- Tangent of θ: $\tan\theta = \dfrac{y}{x}$
- Cotangent of θ: $\cot\theta = \dfrac{x}{y}$



csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A

Evaluating the Trigonometric Functions

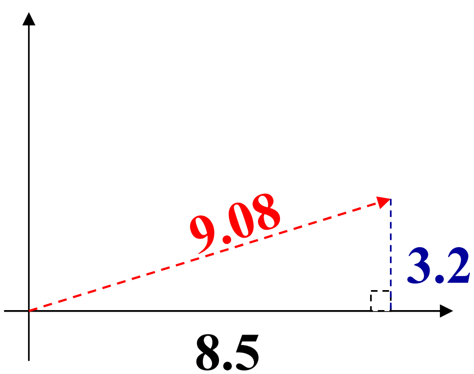
- The values of a trigonometric function are dependent on:
  - The ratios of the sides
  - The Pythagorean Theorem



Example

$\csc A = 1/\sin A$
$\sec A = 1/\cos A$
$\cot A = 1/\tan A$

- Find the values of the trigonometric functions of the angle whose terminal side passes through **(8.5, 3.2)**.

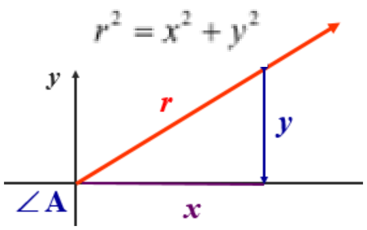


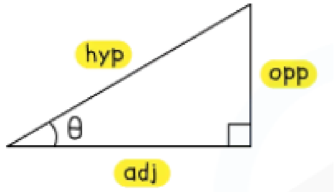
$\sin A = 3.2/9.08 = 0.352$
$\csc A = 1/0.352 = 2.838$
$\cos A = 8.5/9.08 = 0.936$
$\sec A = 1/0.936 = 1.068$
$\tan A = 3.2/8.5 = 0.376$
$\cot A = 1/0.376 = 2.656$

Draw
Label
sohcahtoa
Pythagoras

Given $\cos\theta = 12/13$, find $\sin\theta$ and $\cot\theta$.

csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A





In Exercises 17–24, find the values of the indicated functions. In Exercises 17–20, give answers in exact form. In Exercises 21–24, the values are approximate.

17. Given $\cos\theta = 12/13$, find $\sin\theta$ and $\cot\theta$.

18. Given $\sin\theta = 1/2$, find $\cos\theta$ and $\csc\theta$.

19. Given $\tan\theta = 2$, find $\sin\theta$ and $\sec\theta$.

csc A = 1/sin A

sec A = 1/cos A

cot A = 1/tan A

In Exercises 29–36, answer the given questions.

29. If $\tan\theta = 3/4$, what is the value of $\sin^2\theta + \cos^2\theta$? $[\sin^2\theta = (\sin\theta)^2]$

30. If $\sin\theta = 2/3$, what is the value of $\sec^2\theta - \tan^2\theta$?

31. If $y = \sin\theta$, what is $\cos\theta$ in terms of y?

32. If $x = \cos\theta$, what is $\tan\theta$ in terms of x?

A

17. $\cos\theta = \frac{12}{13} \Rightarrow x = 12$ and $r = 13$ with θ in QL

$r^2 = x^2 + y^2 \Rightarrow 169 = 144 + y^2 \Rightarrow y^2 = 25$

$y = 5$

$\sin\theta = \frac{y}{r} = \frac{5}{13}$, $\cot\theta = \frac{x}{y} = \frac{12}{5}$.

18. $\sin\theta = \frac{1}{2} = \frac{y}{r}$

$r^2 = x^2 + y^2$; $2^2 = x^2 + 1$; $x^2 = 3$; $x = \pm\sqrt{3}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$ for acute θ

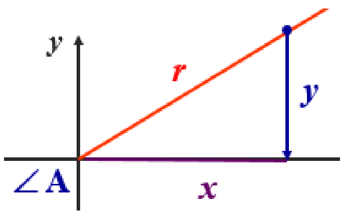
$\csc\theta = \frac{r}{y} = \frac{2}{1} = 2$ for acute θ

19. $\tan\theta = 2 = \frac{y}{x}$

$r^2 = x^2 + y^2 = 1^2 + 2^2 = 5$; $r = \pm\sqrt{5}$

$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{5}}$ for acute θ

$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{1} = \sqrt{5}$ for acute θ



29. $\sin^2\theta + \cos^2\theta = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} = \frac{16}{25}$

$= \frac{25}{25} = 1$

30. $\sin\theta = \frac{y}{r} = \frac{2}{3} \Rightarrow y = 2,\ r = 3$

$x = \sqrt{r^2 - y^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$

$\sec\theta = \frac{r}{x} = \frac{3}{\sqrt{5}}$, $\tan\theta = \frac{y}{x} = \frac{2}{\sqrt{5}}$

$\sec^2 - \tan^2\theta = \left(\frac{3}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = 1$

31. $\sin\theta = \frac{y}{r} = y \Rightarrow r = 1$

$\cos\theta = \frac{x}{r} = x = \sqrt{r^2 - y^2} = \sqrt{1 - \sin^2\theta}$

32. $\cos\theta = \frac{x}{r} = x \Rightarrow r = 1$

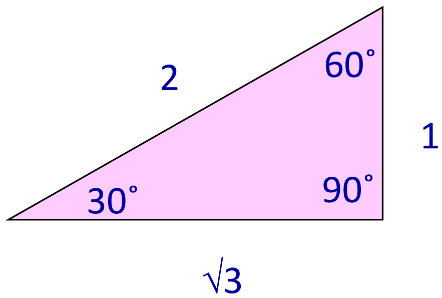
$\tan\theta = \frac{y}{x} = \sqrt{\frac{r^2 - x^2}{x}} = \sqrt{\frac{1 - x^2}{x}}$

Ch. 4.3: Values of the Trigonometric Functions

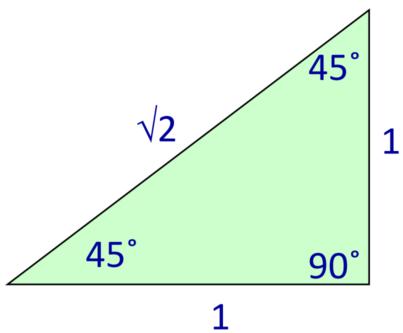
- Using our understanding of the 30°-60°-90° triangle and the 45°-45°-90° triangle, trigonometric ratios can be readily determined.
- Scale drawings of these triangles are used to calculate these ratios.
- It is helpful to be familiar with these values as they commonly occur.

Values of the Trigonometric Functions

- The 30°-60°-90° triangle:



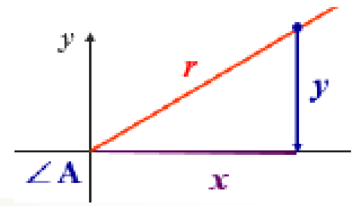
- The 45°-45°-90° triangle:



TASK

Ratios (sin,cos,tan) ?

Finding Unknown Angles



- We determine the unknown angles using the ***inverse*** trigonometric keys of the calculator.
- The notation used for the ratios is:

$$\theta = \sin^{-1}\frac{y}{r} \qquad \theta = \cos^{-1}\frac{x}{r} \qquad \theta = \tan^{-1}\frac{y}{x}$$

- Another commonly used notation is ***arcsin***, ***arccos*** and ***arctan***.

Finding Unknown Angles

- If $\sin A = 0.496$ then $A = \sin^{-1} 0.496$
- *Before you begin, do you want the angle in* ***degrees****? If so, make sure your calculator is on* ***degree*** *mode!*
- $A = \sin^{-1} 0.496 = 29.74°$

- To change the settings for the number of decimal places, the number of significant digits, or the exponential display format, press the MODE key a number of times until you reach the setup screen shown below.

```
Fix Sci Norm
1   2   3
```

Find the missing θ or x in each of the following

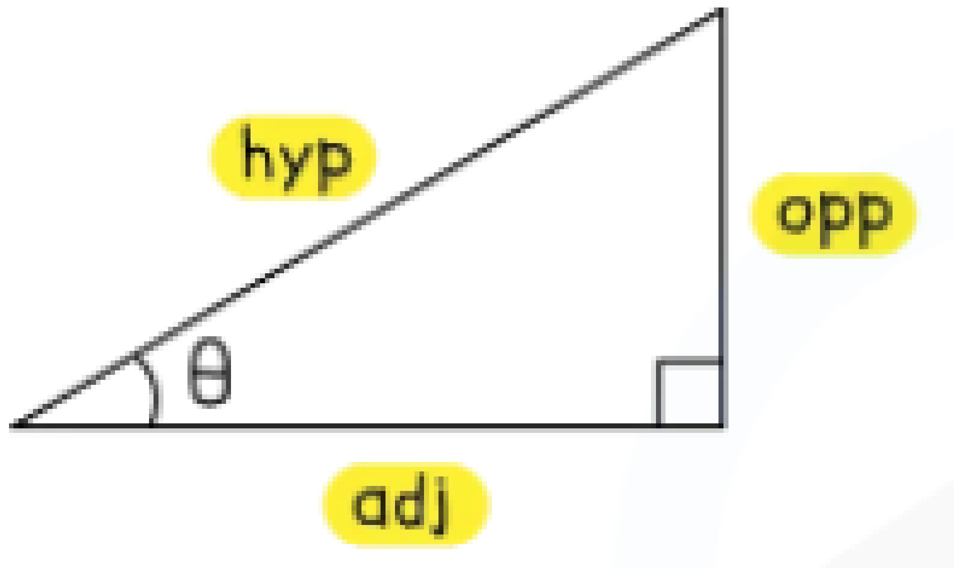
Sin θ = 0.9

Cos 65 = x

Tan θ = 0.4

etc

Ch. 4.4: The Right Triangle



The Trigonometric Functions

- Sine of θ: $$\sin\theta = \frac{\text{side opposite } A}{\text{hypotenuse}}$$
- Cosine of θ: $$\cos\theta = \frac{\text{side adjacent } A}{\text{hypotenuse}}$$
- Tangent of θ: $$\tan\theta = \frac{\text{side opposite } A}{\text{side adjacent } A}$$

SOH CAH TOA

SOH CAH TOA

EXAMPLE 4 Given two sides—find other parts

COMPLETE for ALL sides and angles

.Trig functions
.180º
.Pythagoras

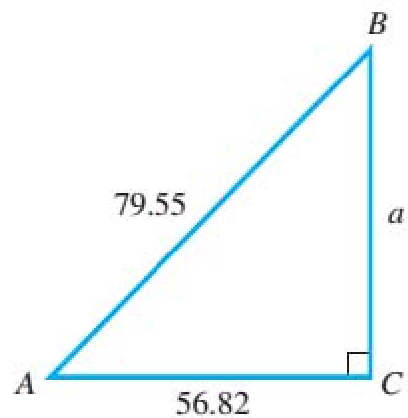


Fig. 39

Solve the right triangle with $b = 56.82$ and $c = 79.55$.

We sketch the right triangle as shown in Fig. 39. Because two sides are given, we will use the Pythagorean theorem to find the third side a. Also, we will use the cosine to find $\angle A$.

Because $c^2 = a^2 + b^2, a^2 = c^2 - b^2$. Therefore,

$$a = \sqrt{c^2 - b^2} = \sqrt{79.55^2 - 56.82^2}$$
$$= 55.67$$

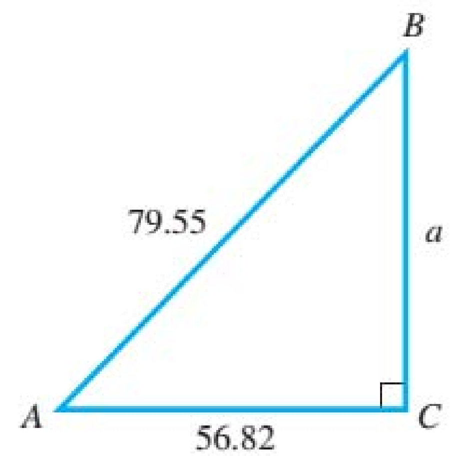
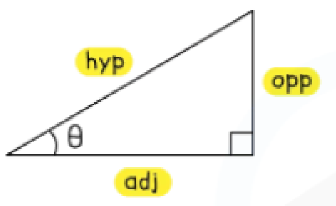
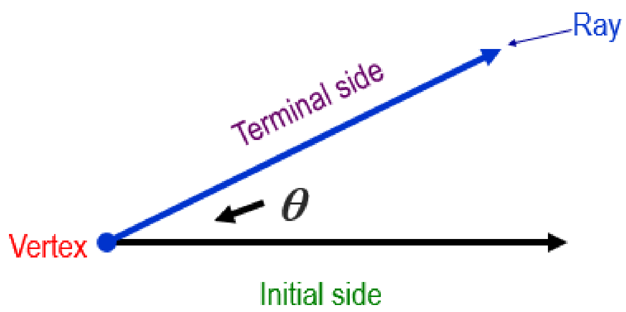


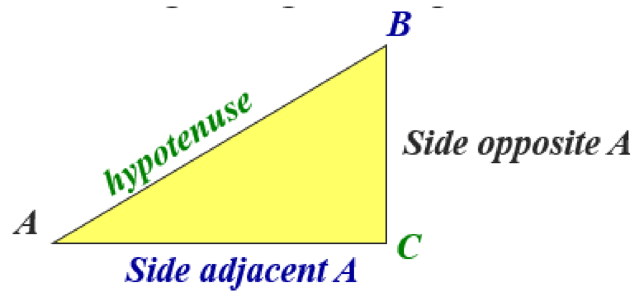
Fig. 39

Examples (given 2 find the rest)





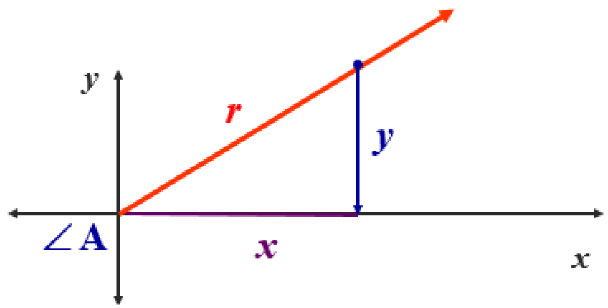
Summary



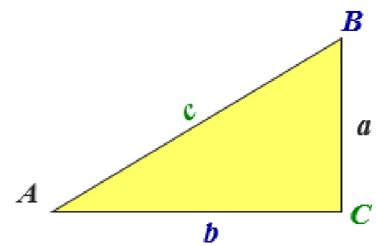
Coterminal
Positive = θ + 360
Negative = θ - 360

Pythagoras

SOH CAH TOA
S=O/H, C=A/H, T=O/A

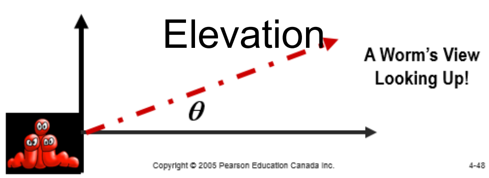


360° is 2π rads

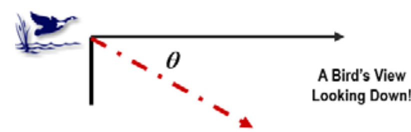


- Sine of θ: $\sin\theta = \frac{y}{r}$
- Cosecant of θ: $\csc\theta = \frac{r}{y}$

Elevation



csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A

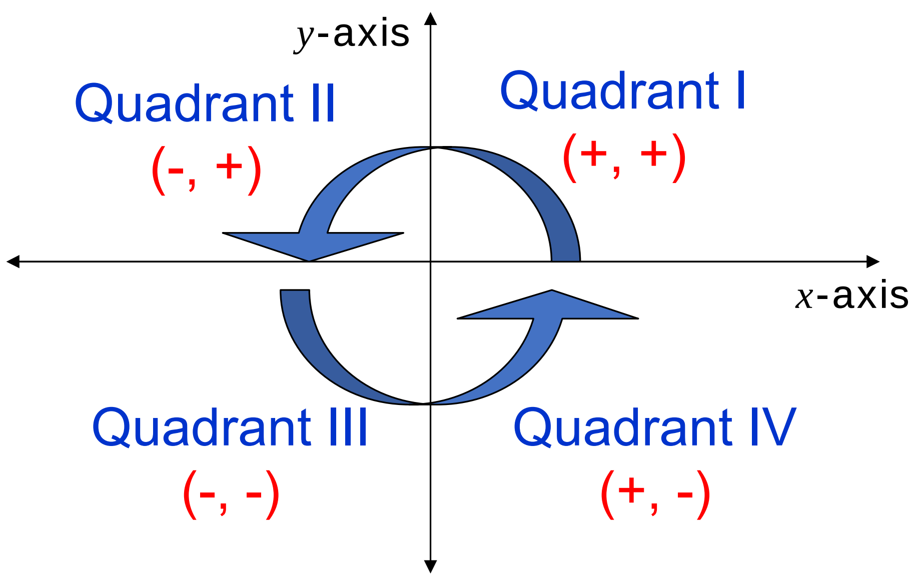


Depression

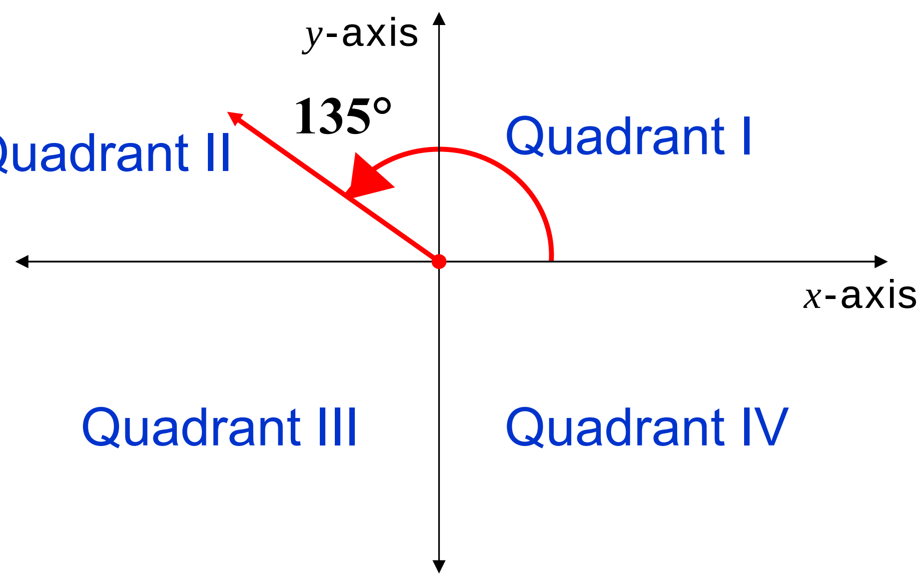
Chapter 8

Trigonometric Functions of Any Angle

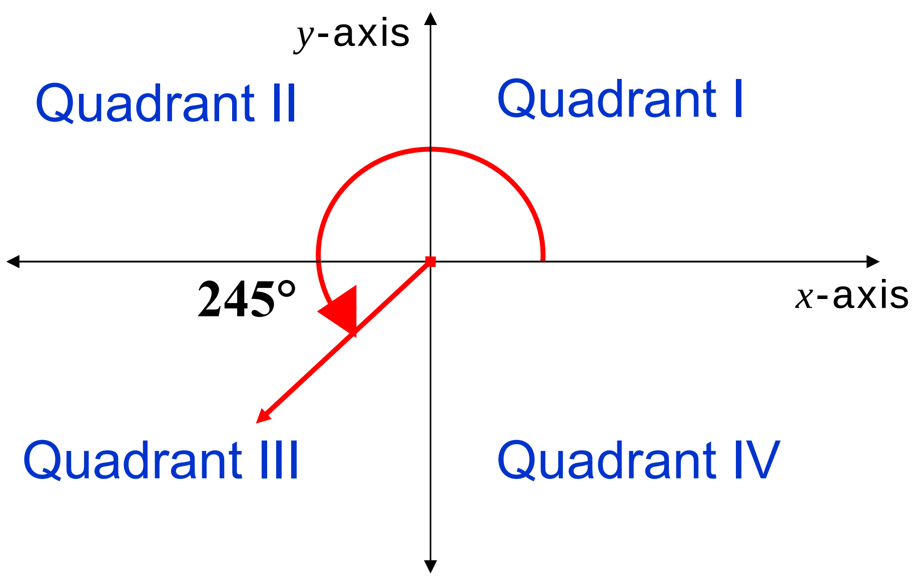
Mapping the Cartesian (*xy*) plane.



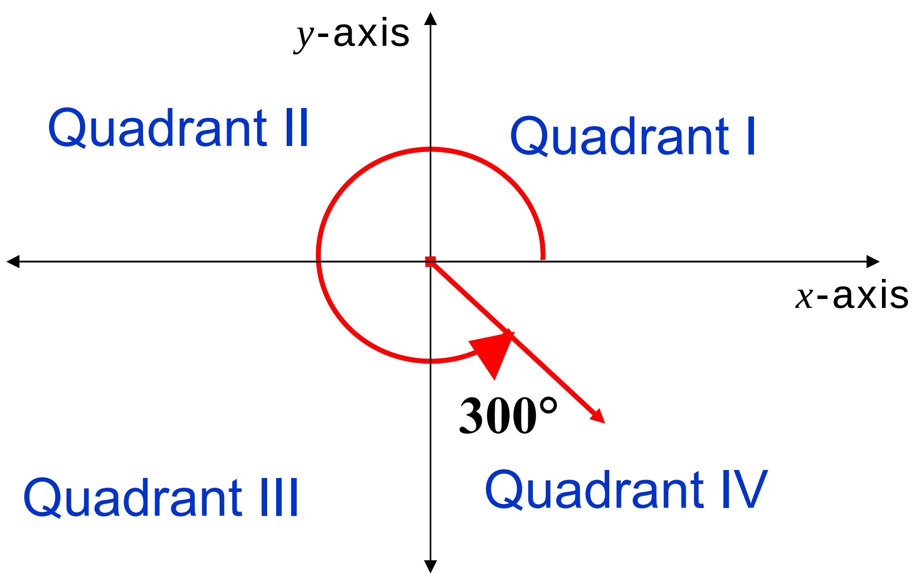
An Angle in QII



An Angle in QIII



An Angle in QIV



The CAST Rule

y-axis

S

QII

Only the **Sine** function is positive.

QI

All trig functions are positive.

A

x-axis

T

QIII

Only the **Tangent** function is positive.

QIV

Only the **Cosine** function is positive.

C

Useful relationships

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

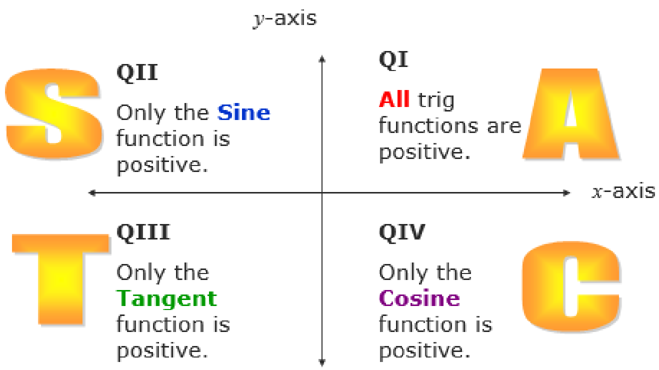
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

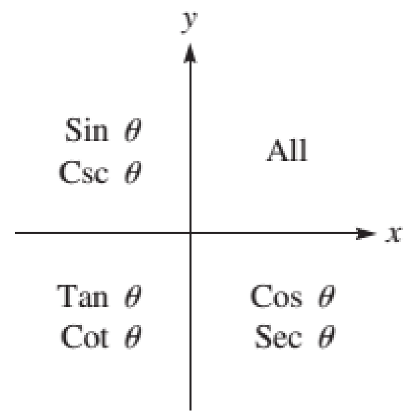
$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$



Is it positive or negative?
Make up angles then ask if functions are positive

csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A



Positive functions

EXAMPLE 4 Positive and negative functions

(a) The following are ***positive:***

$\sin 150°$ $\cos 290°$ $\tan 190°$ $\cot 260°$ $\sec 350°$ $\csc 100°$

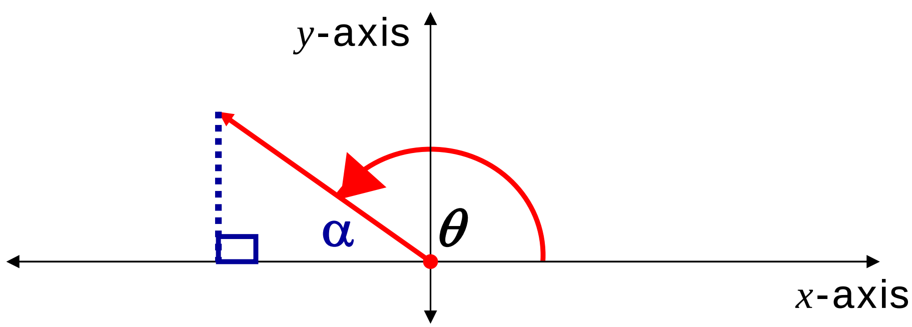
(b) The following are ***negative:***

$\sin 300°$ $\cos 150°$ $\tan 100°$ $\cot 300°$ $\sec 200°$ $\csc 250°$ ■

Ch. 8.2: Trigonometric Functions of Any Angle

- Given an angle in any quadrant, we can determine the trigonometric functions of that angle.
- Given the trigonometric function(s), we can determine the angle in any quadrant.
- When determining the angle in any quadrant, we need to make use of the reference angle.

Reference Angles



- The ***reference angle,*** α, of a given angle is the acute angle formed by the terminal side of the angle and the ***x***-axis.

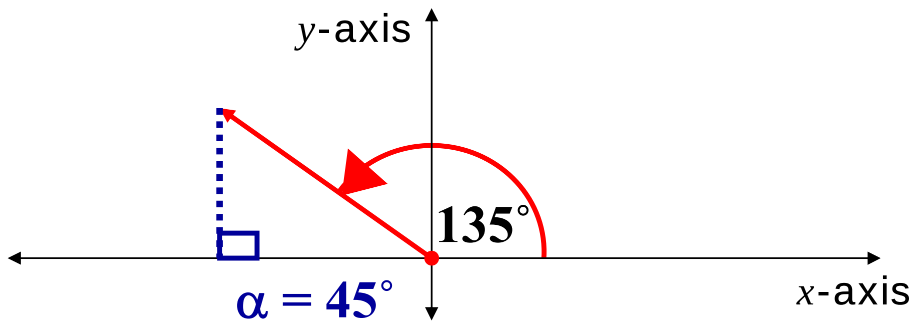
Reference Angles

- The general formula for finding the reference angle is:

$$F(\theta_2) = \pm F\left(180° - \theta_2\right) = \pm F\left(\alpha\right)$$

- The sign used depends on the sign of the function in the second quadrant.
- F represents any trigonometric function

Example



- Angle **45°** is the reference angle to **135°**.

$$F(\theta_2) = \pm F\left(180° - \theta_2\right) = \pm F\left(\alpha\right)$$
$$F(45) = \pm F\left(180° - 135°\right) = \pm F\left(\alpha\right)$$

Steps to Finding Trig Values in any Quadrant

1. Plot the given point.
2. Identify the reference angle & label the sides accordingly.
3. Find the reference angle using $\tan^{-1}$.
4. Add the appropriate degrees to obtain the angle looking for.
5. Either use the calculator to find the trig functions or use the given information. You may have to find the hypotenuse using

$$r = \sqrt{x^2 + y^2}$$

Finding the Reference Angle

Show as Physics

- The required angle θ is found by using the reference angle as shown:

$$\theta = \alpha \text{ (Q I)}$$

$$\theta = 180^\circ - \alpha \text{ (Q II)}$$

$$\theta = 180^\circ + \alpha \text{ (Q III)}$$

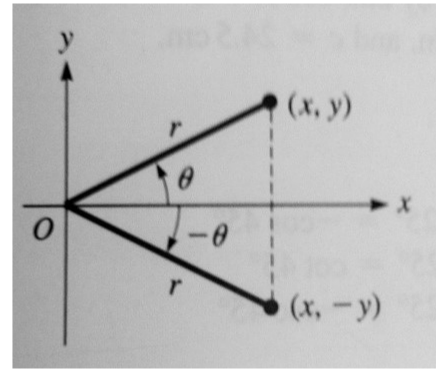
$$\theta = 360^\circ - \alpha \text{ (Q IV)}$$

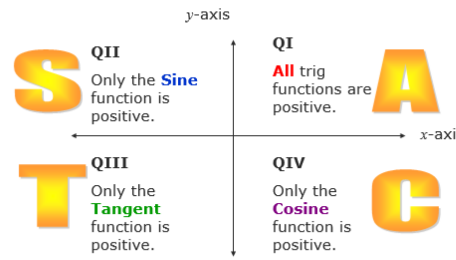
The Quadrantal Angles

- These angles are represented by the x and y axes.
- They are $0°$, $90°$, $180°$, $270°$ and $360°$.
- Their terminal side lies on a coordinate axis.

Negative Angles

- To find the values of the functions of negative angles, we can use functions of corresponding positive angles, ***if we use the correct sign***.





$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

csc(θ) = 1/sin(θ)	sec(θ) = 1/cos(θ)	cot(θ) = 1/tan(θ)

EXAMPLE 1 Coterminal angles

The following pairs of angles are coterminal.

$$390° \text{ and } 30° \qquad 900° \text{ and } 180° \qquad -150° \text{ and } 210° \text{ (see Fig. 5)}$$

The trigonometric functions of both angles in each pair are equal. For example,

$$\sin 390° = \sin 30° \quad \text{and} \quad \tan(-150°) = \tan 210°$$

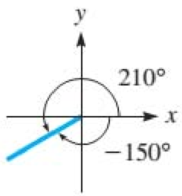


Fig. 5

EXAMPLE 2 Quadrant II reference angles

In Fig. 6, the trigonometric functions of θ are as follows:

$$\sin\theta_2 = \sin(180° - \theta_2) = \sin\alpha = \sin\theta_1 = \tfrac{4}{5} = 0.8000$$

$$\cos\theta_2 = -\cos\theta_1 = -\tfrac{3}{5} = -0.6000 \qquad \tan\theta_2 = -\tfrac{4}{3} = -1.333$$

$$\cot\theta_2 = -\tfrac{3}{4} = -0.7500 \qquad \sec\theta_2 = -\tfrac{5}{3} = -1.667 \qquad \csc\theta_2 = \tfrac{5}{4} = 1.250$$

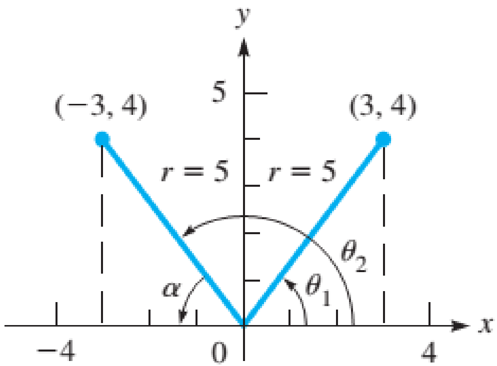


Fig. 6

Ch. 8.3: Radians

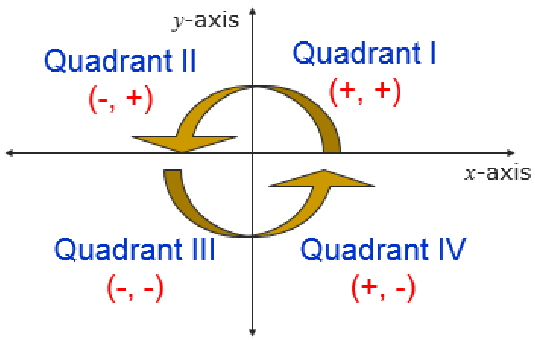
360° is 2π rads

Or use conversion factors

- Procedure for Converting Angle Measurements
  - To convert an angle measured in degrees to the same angle measured in radians, ***multiply the number of degrees by*** π rads/180°.
  - To convert an angle measured in radians to the same angle measured in degrees, ***multiply the number of radians by*** 180°/π rads.

$360°$ is 2π rads

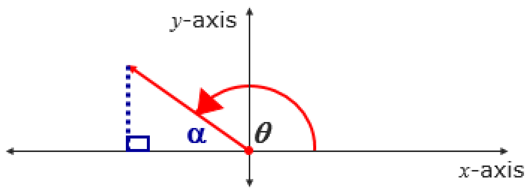
Deg to rad
Rad to deg



$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

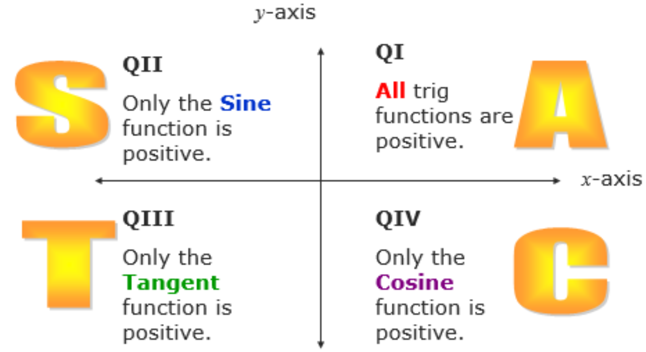


- The ***reference angle, α***, of a given angle is the acute angle formed by the terminal side of the angle and the x-axis.

$$2\pi \ radians = 360 \ degrees$$

$$1 \, radian = \frac{180}{\pi} \ degrees$$

$$1 \, degree = \frac{\pi}{180} \ radians$$



$\theta = \alpha$ (Q I)

$\theta = 180^\circ - \alpha$ (Q II)

$\theta = 180^\circ + \alpha$ (Q III)

$\theta = 360^\circ - \alpha$ (Q IV)

Chapter 10

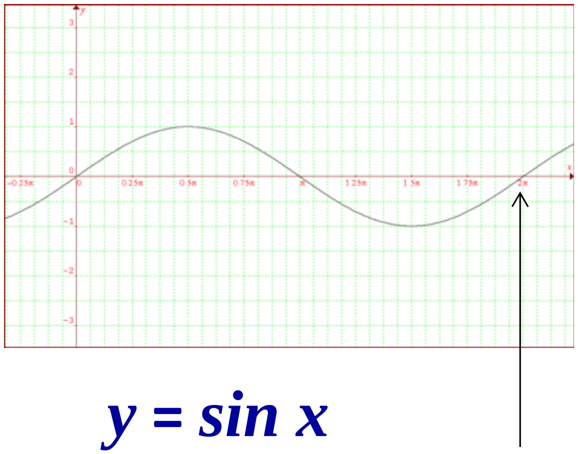
Graphing the Trigonometric Functions

Ch. 10.1: Graphs of $y = a\sin x$ & $y = a\cos x$

- The ***angles*** of the graphs of the trigonometric functions **will be expressed** in ***radians*****.**
- In this way, both the independent variable and the dependent variable are real numbers.

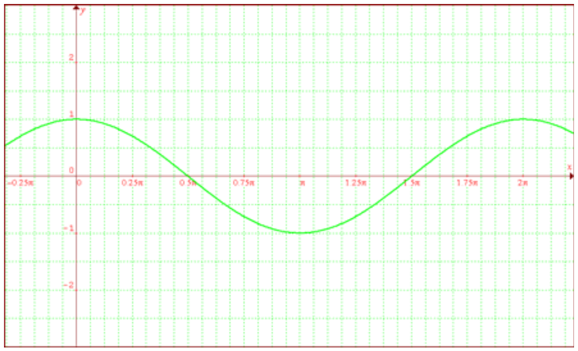
The Periodic Curve

- $y = \sin x$ and $y = \cos x$ are known as sinusoidal or periodic functions.



$y = \sin x$

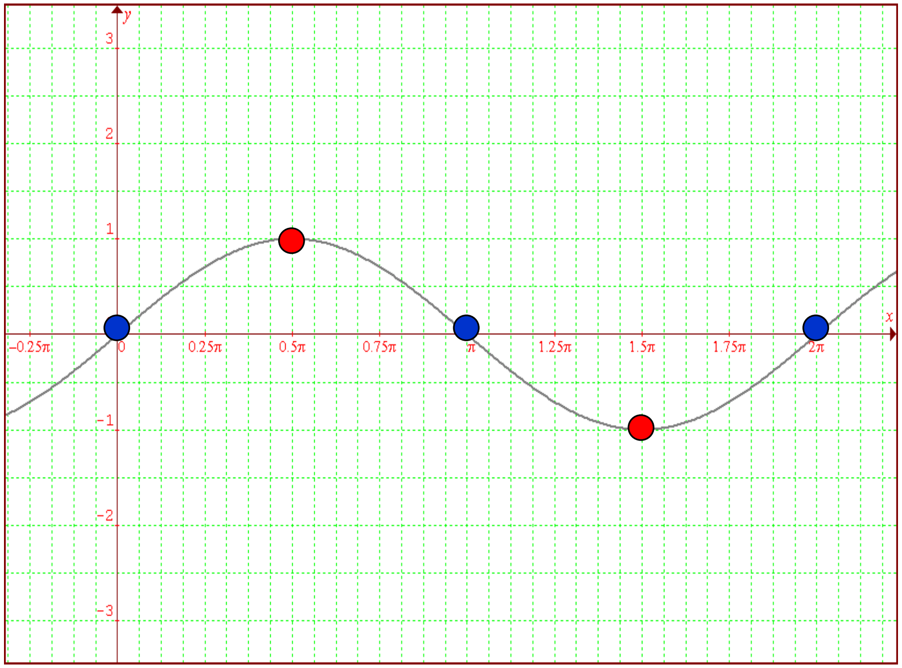
2π



$y = \cos x$

Key Points on the Normal Sine Curve

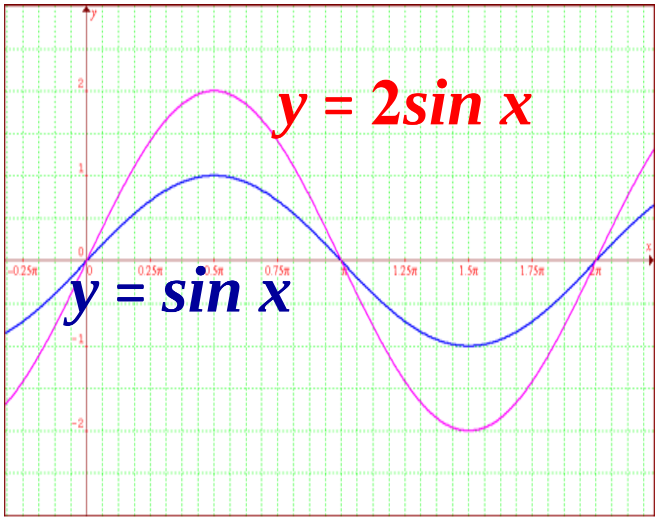
1. 1^{st} intercept: $(0, 0)$
2. 2^{nd} intercept: $(\pi, 0)$
3. 3^{rd} intercept: $(2\pi, 0)$
4. Maximum: $(\pi/2, 1)$
5. Minimum: $(3\pi/2, -1)$



Amplitude

- All y-values obtained for the graph of $y = \sin x$ and $y = \cos x$ are to be multiplied by a.
- The amplitude increases by $|a|$ amount.

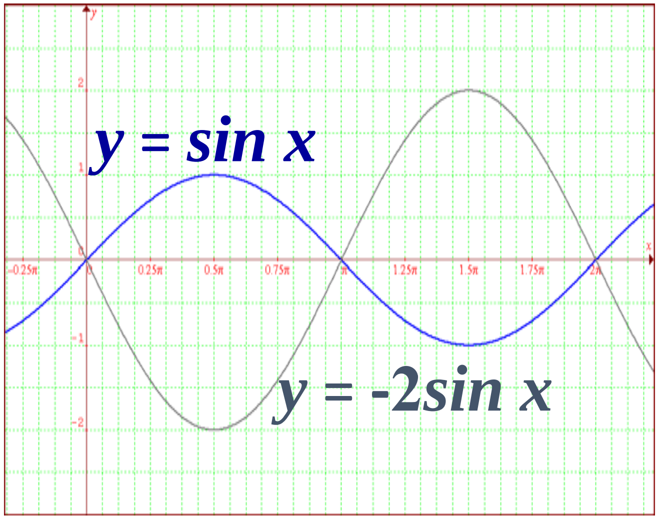
Amplitude



- Note the location of the 5 key points used in graphing.

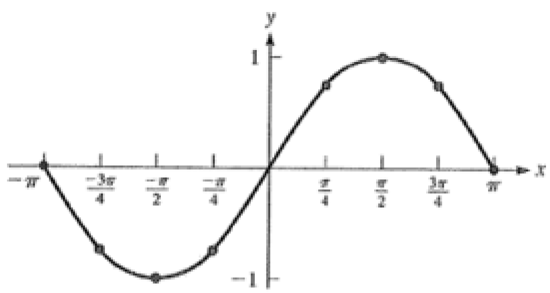
Amplitude

- The negative reverses the direction of the graph.
- The intercepts remain the same but the maximum & minimum are reversed.



3. $y = \sin x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0

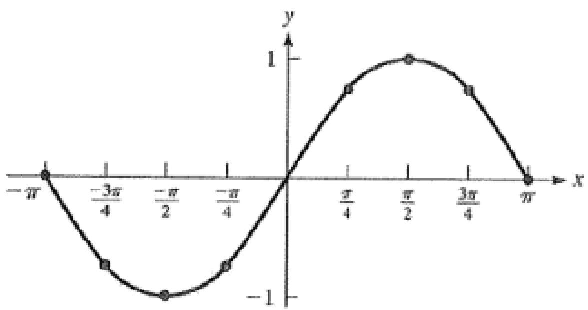


Do (plot –π to π)
y=sinx
y=3cosx
y=cosx
y=-4sinx

A

3. $y = \sin x$

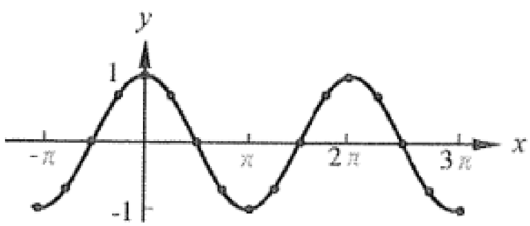
x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0



4. $y = \cos x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	-1	-0.7	0	0.7	1	0.7	0	-0.7	-1

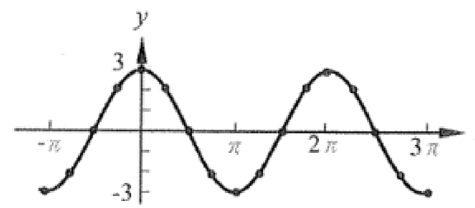
x	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-0.7	0	0.7	1	0.7	0	-0.7	-1



5. $y = 3 \cos x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	-3	-2.1	0	2.1	3	2.1	0	-2.1	3

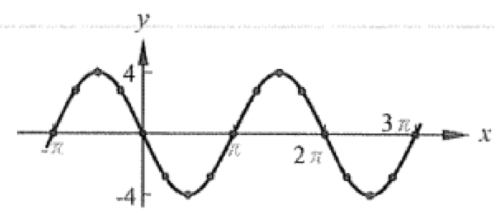
x	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-2.1	0	2.1	3	2.1	0	-2.1	-3



6. $y = -4 \sin x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	2.8	4	2.8	0	-2.8	-4	-2.8	0

x	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	2.8	4	2.8	0	-2.8	-4	-2.8	0

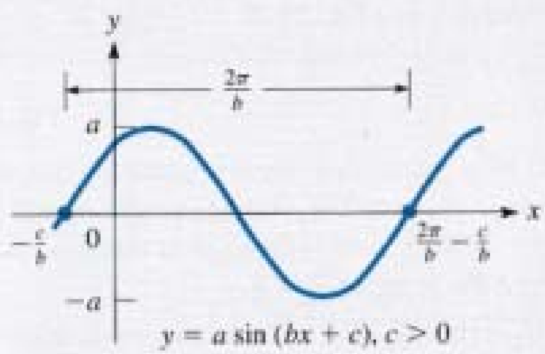


For this semester you just need to know this for multiple choice questions

CHAPTER 10 EQUATIONS

For the graphs of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$

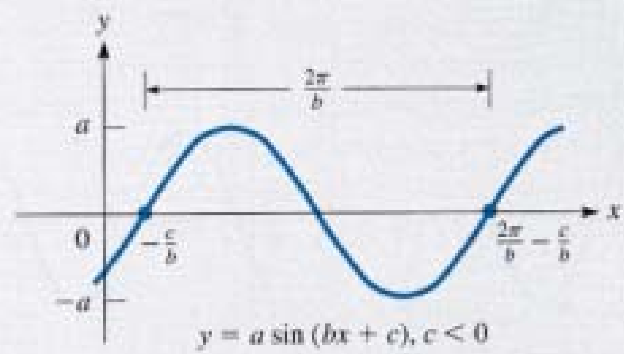
Amplitude $= |a|$ Period $= \frac{2\pi}{b}$ Displacement $= -\frac{c}{b}$ **(10.1)**



$y = a\sin(bx + c), c > 0$

(a)

For each
$a > 0, b > 0$



$y = a\sin(bx + c), c < 0$

(b)

Reciprocal relationships $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$ **(10.2)**

Ch. 10.2: Graphs of $y = a\sin bx$ & $y = a\cos bx$

- When graphing $y = \sin x$ and $y = \cos x$ the values of y repeat every 2π radians.
- We say that these functions are periodic and have a period of 2π rads.
- Graphing $y = a\sin bx$ or $y = a\cos bx$
- **Period = $2\pi/b$**
- We still use the 5 key values to sketch a trigonometric function to include its period.

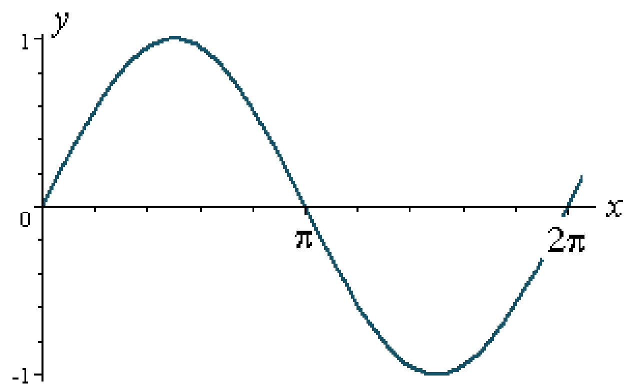
$$y = a\sin bx$$

Period = 2π/b

a = amplitude

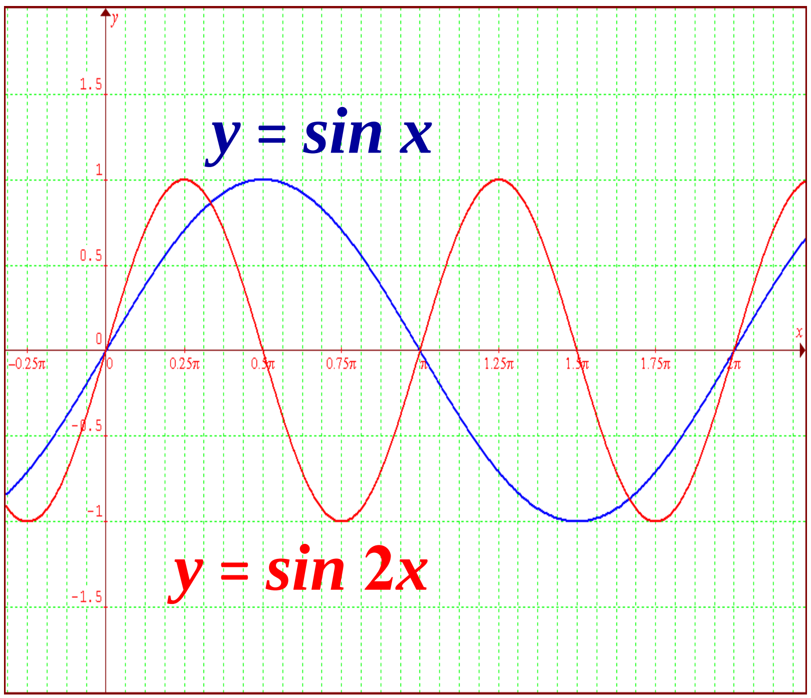
We start with $y = \sin x$.

It has **amplitude** $= 1$ and **period** $= 2\pi$.



Period

$y = asinbx$ **Period = 2π/b**



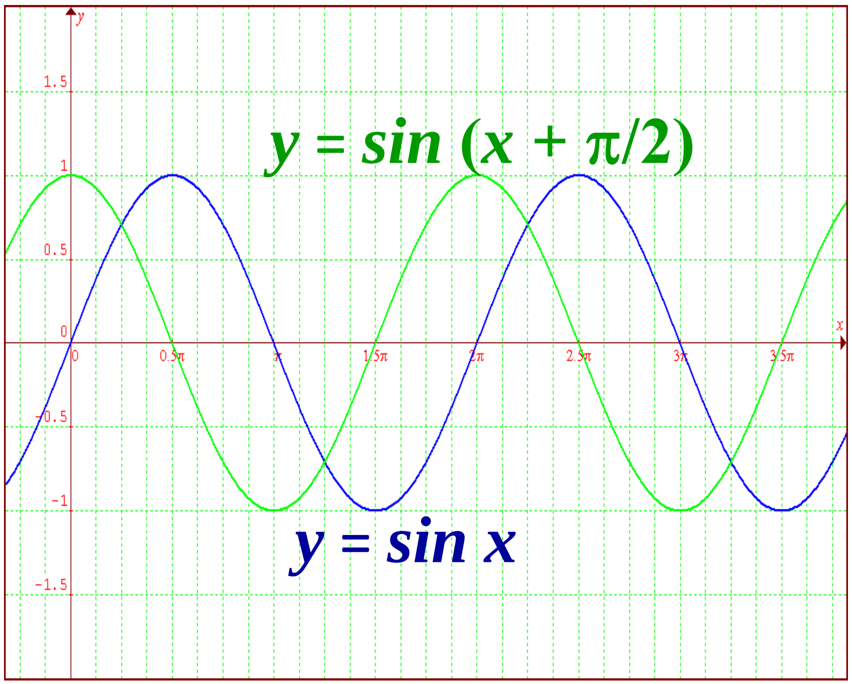
- There are twice as many cycles.
- Once again, note the location of the 5 key points used in graphing.

Ch. 10.3: Graphs of $y = a\sin(bx+c)$ & $y = a\cos(bx+c)$

- The effect of c in the equations $y = a\sin(bx+c)$ and $y = a\cos(bx+c)$
  - shift the curve to the ***left*** if $c > 0$, or
  - shift the curve to the ***right*** if $c < 0$.
- **Displacement or Phase Shift:**
  - The amount of shift is given by $-c/b$.

Displacement

***y* = *a*sin(*bx*+*c*) & *y* = *a*cos(*bx*+*c*)**



- The displacement is ***negative*** (to the ***left***) for $c > 0$.

Learn this

- BE ABLE TO FIND THE AMPLITUDE PERIOD AND DISPLACEMENT

$$y = a \sin(bx + c) \quad \& \quad y = a \cos(bx + c):$$

- Amplitude = $|a|$
- Period = $2\pi/b$
- Displacement = $-c/b$

Worked examples

4.1 1,2,3,4

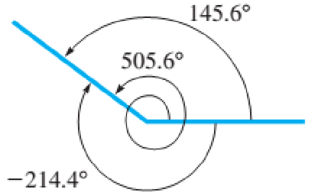


Fig. 3

EXAMPLE 1 Coterminal angles

Determine the measures of two angles that are coterminal with an angle of 145.6°.

Because there are 360° in one complete rotation, we can find a coterminal angle by adding 360° to the given angle of 145.6° to get 505.6°. Another coterminal angle can be found by subtracting 360° from 145.6° to get −214.4°. See the angles in Fig. 3. We could continue to add 360°, or subtract 360°, as many times as needed to get as many additional coterminal angles as may be required. ■

The Trigonometric Functions

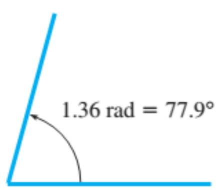


Fig. 4

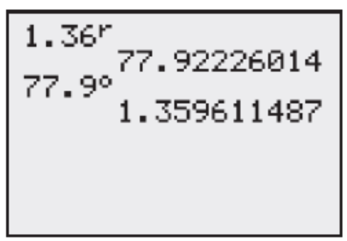


Fig. 5

EXAMPLE 2 Convert radians to degrees

Express 1.36 rad in degrees.

We know that π rad = 180°, which means 1 rad = $180°/\pi$. Therefore,

$$1.36 \text{ rad} = 1.36\left(\frac{180°}{\pi}\right) = 77.9° \quad \text{to nearest } 0.1°$$

This angle is shown in Fig. 4. We again note that degrees and radians are simply two different ways of measuring an angle.

In Fig. 5, a calculator display shows the conversions of 1.36 rad to degrees (calculator in *degree* mode) and 77.9° to radians (calculator in *radian* mode). ■

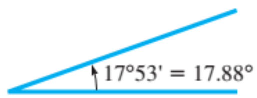
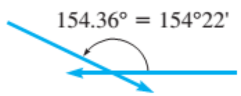


Fig. 6



EXAMPLE 3 Degrees, minutes—decimal form

(a) We change 17°53′ to decimal form by using the fact that $1° = 60'$. This means that $53' = \left(\frac{53}{60}\right)^{\circ} = 0.88°$ (to nearest 0.01°). Therefore, $17°563' = 17.88°$. See Fig. 6.

(b) The angle between two laser beams is 154.36°. To change this to an angle measured to the nearest minute, we have

$$0.36° = 0.36(60') = 22'$$

This means that $154.36° = 154°22'$. See Fig. 7. ■

EXAMPLE 4 Angles in standard position

A standard position angle of 60° is a first-quadrant angle with its terminal side 60° from the x-axis. See Fig. 8(a).

A second-quadrant angle of 130° is shown in Fig. 8(b).

A third-quadrant angle of 225° is shown in Fig. 8(c).

A fourth-quadrant angle of 340° is shown in Fig. 8(d).

A standard-position angle of −120° is shown in Fig. 8(e). Because the terminal side is in the third quadrant, it is a third-quadrant angle.

A standard-position angle of 90° is a quadrantal angle since its terminal side is the positive y-axis. See Fig. 8(f).

Practice Exercise

3. In Fig. 8, which terminal side is that of a standard position angle of 240°?

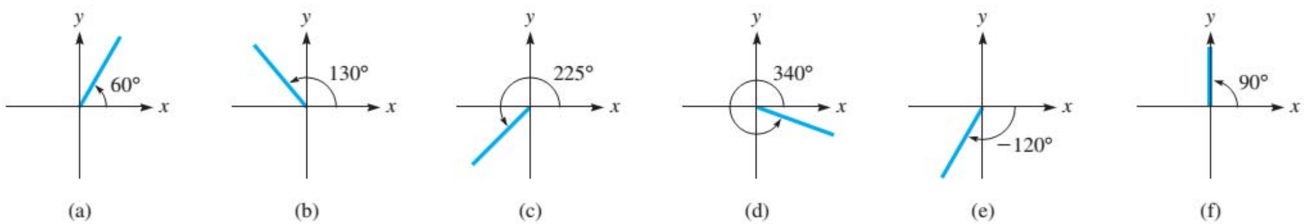


Fig. 8

4.2 1,3,5

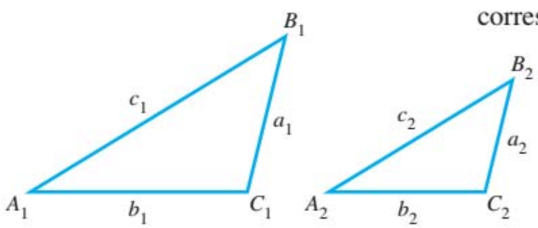


Fig. 10

EXAMPLE 1 Similar triangles

In Fig. 10, the triangles are similar and are lettered so that corresponding sides and angles have the same letters. That is, angles A_1 and A_2, angles B_1 and B_2, and angles C_1 and C_2 are pairs of corresponding angles. The pairs of corresponding sides are a_1 and a_2, b_1 and b_2, and c_1 and c_2. From the properties of similar triangles, we know that the corresponding angles are equal, or

$$\angle A_1 = \angle A_2 \qquad \angle B_1 = \angle B_2 \qquad \angle C_1 = \angle C_2$$

Also, the corresponding sides are proportional, which we can show as

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \qquad \frac{a_1}{a_2} = \frac{c_1}{c_2} \qquad \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

In Example 1, if we multiply both sides of $a_1/a_2 = b_1/b_2$ by a_2/b_1, we get

$$\frac{a_1}{a_2}\left(\frac{a_2}{b_1}\right) = \frac{b_1}{b_2}\left(\frac{a_2}{b_1}\right), \quad \text{or} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

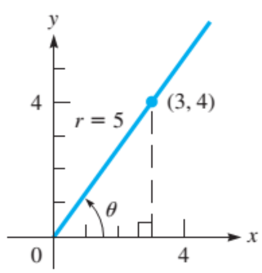


Fig. 14

EXAMPLE 3 Values of trigonometric functions

Find the values of the trigonometric functions of the standard-position angle θ with its terminal side passing through the point $(3, 4)$.

By placing the angle in standard position, as shown in Fig. 14, and drawing the terminal side through $(3, 4)$, we find by use of the Pythagorean theorem that

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Using the values $x = 3, y = 4$, and $r = 5$, we find that

$$\sin\theta = \frac{4}{5} \qquad \cos\theta = \frac{3}{5} \qquad \tan\theta = \frac{4}{3}$$

$$\cot\theta = \frac{3}{4} \qquad \sec\theta = \frac{5}{3} \qquad \csc\theta = \frac{5}{4}$$

We have left each of these results in the form of a fraction, which is considered to be an *exact form* in that there has been no approximation made. In writing decimal values, we find that $\tan\theta = 1.333$ and $\sec\theta = 1.667$, where these values have been rounded off and are therefore *approximate*. ■

Practice Exercise

1. In Example 3, change (3, 4) to (4, 3) and then find $\tan\theta$ and $\sec\theta$.

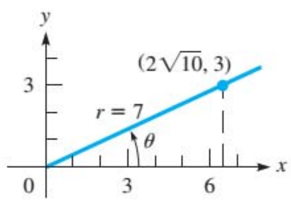


Fig. 17

Practice Exercise

2. In Example 5, change $\sin\theta = 3/7$ to $\cos\theta = 3/7$, and then find approximate values of $\sin\theta$ and $\cot\theta$.

EXAMPLE 5 Given one function—find others

If we know that $\sin\theta = 3/7$, and that θ is a first-quadrant angle, we know the ratio of the ordinate to the radius vector (y to r) is 3 to 7. Therefore, the point on the terminal side for which $y = 3$ can be found by use of the Pythagorean theorem. The x-value for this point is

$$x = \sqrt{7^2 - 3^2} = \sqrt{49 - 9} = \sqrt{40} = 2\sqrt{10}$$

Therefore, the point $(2\sqrt{10}, 3)$ is on the terminal side, as shown in Fig. 17.

Therefore, using the values $x = 2\sqrt{10}$, $y = 3$, and $r = 7$, we have the other trigonometric functions of θ. They are

$$\cos\theta = \frac{2\sqrt{10}}{7} \quad \tan\theta = \frac{3}{2\sqrt{10}} \quad \cot\theta = \frac{2\sqrt{10}}{3} \quad \sec\theta = \frac{7}{2\sqrt{10}} \quad \csc\theta = \frac{7}{3}$$

These values are *exact*. *Approximate* decimal values found on a calculator are

$$\cos\theta = 0.9035 \quad \tan\theta = 0.4743 \quad \cot\theta = 2.108$$
$$\sec\theta = 1.107 \quad \csc\theta = 2.333$$

4.3 1,2,4,5,7

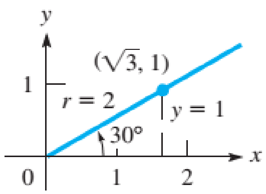


Fig. 18

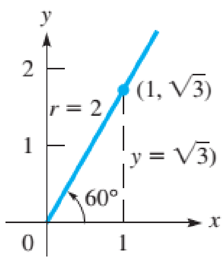


Fig. 19

EXAMPLE 1 Function values of 30° and 60°

From geometry, we find that in a right triangle, the side opposite a 30° angle is one-half of the hypotenuse. Therefore, in Fig. 18, letting $y = 1$ and $r = 2$, and using the Pythagorean theorem, $x = \sqrt{2^2 - 1^2} = \sqrt{3}$. Now with $x = \sqrt{3}$, $y = 1$, and $r = 2$,

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

Using this same method, we find the functions of 60° to be (see Fig. 19.)

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

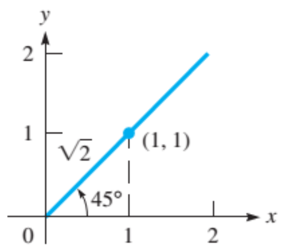


Fig. 20

EXAMPLE 2 Function values of 45°

Find sin 45°, cos 45°, and tan 45°.

If we place an isosceles right triangle with one of its 45° angles in standard position and hypotenuse along the radius vector (see Fig. 20), the terminal side passes through (1, 1), since the legs of the triangle are equal. Using this point, $x = 1, y = 1$, and $r = \sqrt{2}$. Thus,

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = 1$$

In Examples 1 and 2, we have given *exact* values. Decimal approximations are also given in the following table that summarizes the results for 30°, 45°, and 60°.

■ It is helpful to be familiar with these values, as they are used in later sections.

Trigonometric Functions of 30°, 45°, and 60°

	(exact values)			*(decimal approximations)*		
θ	30°	45°	60°	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0.500	0.707	0.866
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0.866	0.707	0.500
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	0.577	1.000	1.732

EXAMPLE 4 Inverse function value from calculator

If $\cos\theta = 0.3527$, which means that $\theta = \cos^{-1} 0.3527$ (θ is the angle whose cosine is 0.3527), we can use a calculator to find θ. The display is shown in Fig. 22.

Therefore, $\theta = 69.35°$ (rounded off). ■

EXAMPLE 5 Reciprocal function value from calculator

To find the value of sec 27.82°, we use the fact that

$$\sec 27.82° = \frac{1}{\cos 27.82°} \quad \text{or} \quad \sec 27.82° = (\cos 27.82°)^{-1}$$

Therefore, we are to find the reciprocal of the value of cos 27.82°. It can be found using either the first two lines, or the third and fourth lines, of the calculator display in Fig. 23. From this display, $\sec 27.82° = 1.131$, rounded off according to the above table.

Note that we calculated the reciprocal $(\cos 27.82°)^{-1}$, and not the angle that would be denoted by using the $\cos^{-1}$ notation. ■

EXAMPLE 7 Given one function—find another

Find $\sin\theta$ if $\sec\theta = 2.504$.

Since the value of $\sec\theta$ is known, we know that $\cos\theta = 2.504^{-1}$ (or $1/2.504$). This in turn tells us that $\theta = \cos^{-1}(2.504^{-1})$. Because we are to find the value of $\sin\theta$, we can see that

$$\sin\theta = \sin(\cos^{-1}(2.504^{-1}))$$

Therefore, we have the calculator display shown in Fig. 24.

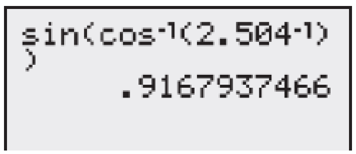


Fig. 24

This means that $\sin\theta = 0.9168$ (rounded off). ■

4.4 1,2,3,4

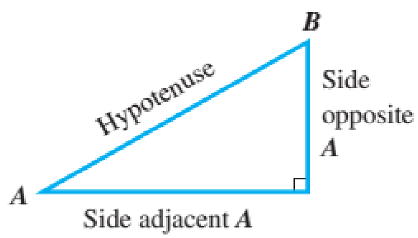


Fig. 34

$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} \qquad \cos A = \frac{\text{side adjacent A}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent } A} \qquad \cot A = \frac{\text{side adjacent } A}{\text{side opposite } A}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent } A} \qquad \csc A = \frac{\text{hypotenuse}}{\text{side opposite } A}$$

EXAMPLE 2 Cofunctions of complementary angles

Given $a = 4$, $b = 7$, and $c = \sqrt{65}$ (see Fig. 35), find sin A, cos A, tan A, sin B,cos B, and tan B in exact form and in approximate decimal form (to three significant digits).

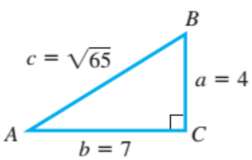


Fig. 35

$$\sin A = \frac{\text{side opposite angle } A}{\text{hypotenuse}} = \frac{4}{\sqrt{65}} = 0.496 \qquad \sin B = \frac{\text{side opposite angle } B}{\text{hypotenuse}} = \frac{7}{\sqrt{65}} = 0.868$$

$$\cos A = \frac{\text{side adjacent angle } A}{\text{hypotenuse}} = \frac{7}{\sqrt{65}} = 0.868 \qquad \cos B = \frac{\text{side adjacent angle } B}{\text{hypotenuse}} = \frac{4}{\sqrt{65}} = 0.496$$

$$\tan A = \frac{\text{side opposite angle } A}{\text{side adjacent angle } A} = \frac{4}{7} = 0.571 \qquad \tan B = \frac{\text{side opposite angle } B}{\text{side adjacent angle } B} = \frac{7}{4} = 1.75$$

We see that A and B are complementary angles. Comparing values of the functions of angles A and B, we see that $\sin A = \cos B$ and $\cos A = \sin B$. ■

EXAMPLE 3 Given angle and side—find other parts

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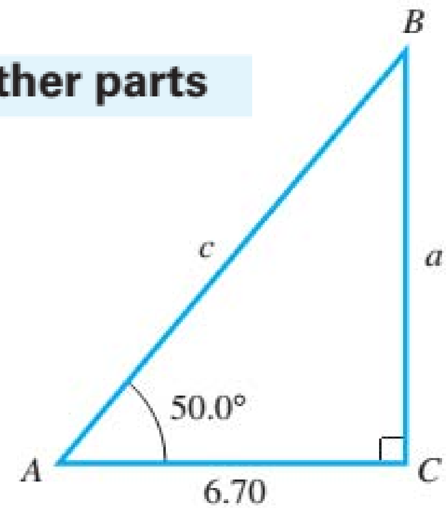


Fig. 36

PROCEDURE FOR SOLVING A RIGHT TRIANGLE

1. *Sketch a right triangle and label the known and unknown sides and angles.*
2. *Express each of the three unknown parts in terms of the known parts and solve for the unknown parts.*
3. *Check the results.* The sum of the angles should be 180°. If only one side is given, check the computed side with the Pythagorean theorem. If two sides are given, check the angles and computed side by using appropriate trigonometric functions.

EXAMPLE 4 Given two sides—find other parts

COMPLETE for ALL
sides and angles

.Trig functions
.180º
.Pythagoras

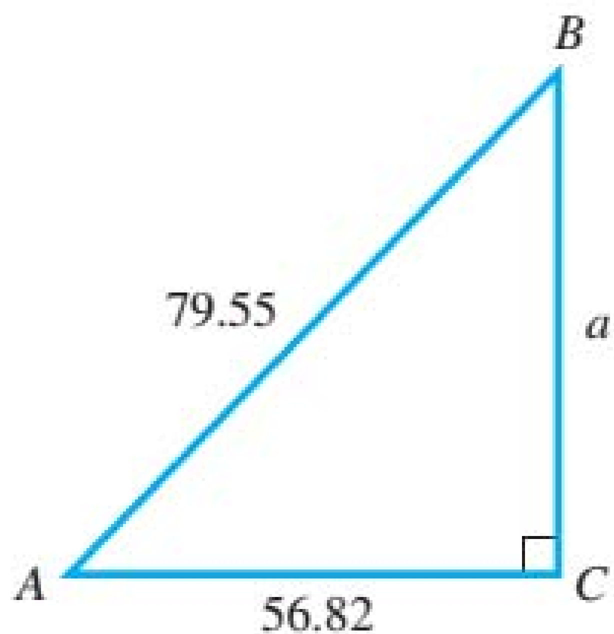


Fig. 39

Solve the right triangle with $b = 56.82$ and $c = 79.55$.

We sketch the right triangle as shown in Fig. 39. Because two sides are given, we will use the Pythagorean theorem to find the third side a. Also, we will use the cosine to find $\angle A$.

Because $c^2 = a^2 + b^2, a^2 = c^2 - b^2$. Therefore,

$$a = \sqrt{c^2 - b^2} = \sqrt{79.55^2 - 56.82^2}$$
$$= 55.67$$

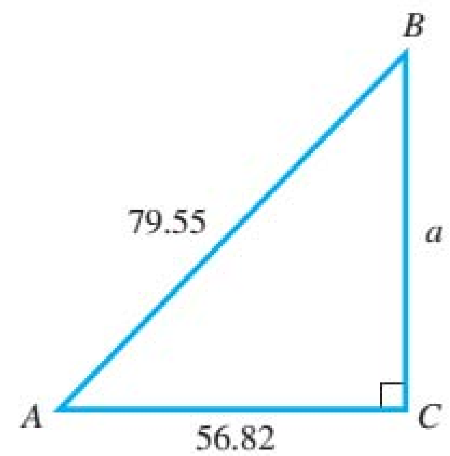
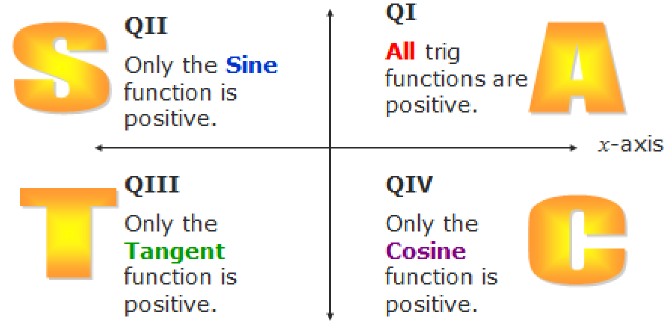


Fig. 39

8.1 1,2,3,4



EXAMPLE 1 Sign of $\sin\theta$ in each quadrant

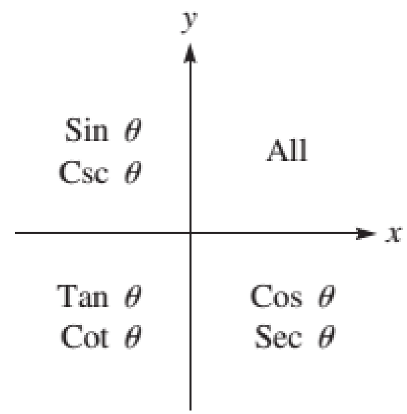
The value of $\sin 20°$ is positive, because the terminal side of 20° is in the first quadrant. The value of $\sin 160°$ is positive, because the terminal side of 160° is in the second quadrant. The values of $\sin 200°$ and $\sin 340°$ are negative, because the terminal sides are in the third and fourth quadrants, respectively. ■

EXAMPLE 2 Sign of $\tan\theta$ in each quadrant

The values of $\tan 20°$ and $\tan 200°$ are positive, because the terminal sides of these angles are in the first and third quadrants, respectively. The values of $\tan 160°$ and $\tan 340°$ are negative, because the terminal sides of these angles are in the second and fourth quadrants, respectively. ■

EXAMPLE 3 Sign of $\cos\theta$ in each quadrant

The values of $\cos 20°$ and $\cos 340°$ are positive, because the terminal sides of these angles are in the first and fourth quadrants, respectively. The values of $\cos 160°$ and $\cos 200°$ are negative, because the terminal sides of these angles are in the second and third quadrants, respectively. ■



Positive functions

EXAMPLE 4 Positive and negative functions

(a) The following are ***positive:***

$$\sin 150° \quad \cos 290° \quad \tan 190° \quad \cot 260° \quad \sec 350° \quad \csc 100°$$

(b) The following are ***negative:***

$$\sin 300° \quad \cos 150° \quad \tan 100° \quad \cot 300° \quad \sec 200° \quad \csc 250°$$

8.2 1,2,3,4,5,8,11

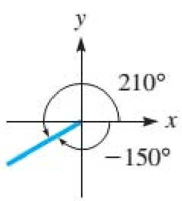


Fig. 5

EXAMPLE 1 Coterminal angles

The following pairs of angles are coterminal.

$$390^\circ \text{ and } 30^\circ \qquad 900^\circ \text{ and } 180^\circ \qquad -150^\circ \text{ and } 210^\circ \text{ (see Fig. 5)}$$

The trigonometric functions of both angles in each pair are equal. For example,

$$\sin 390^\circ = \sin 30^\circ \quad \text{and} \quad \tan(-150^\circ) = \tan 210^\circ$$

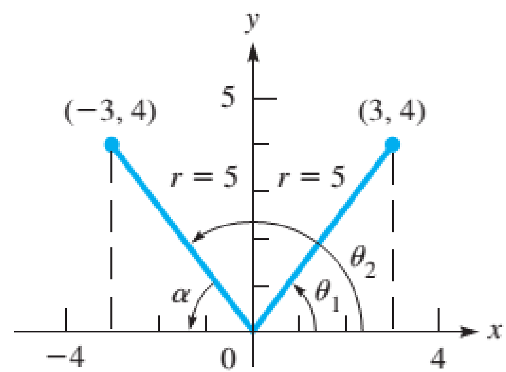


Fig. 6

EXAMPLE 2 Quadrant II reference angles

In Fig. 6, the trigonometric functions of θ are as follows:

$$\sin\theta_2 = \sin(180^\circ - \theta_2) = \sin\alpha = \sin\theta_1 = \tfrac{4}{5} = 0.8000$$

$$\cos\theta_2 = -\cos\theta_1 = -\tfrac{3}{5} = -0.6000 \qquad \tan\theta_2 = -\tfrac{4}{3} = -1.333$$

$$\cot\theta_2 = -\tfrac{3}{4} = -0.7500 \qquad \sec\theta_2 = -\tfrac{5}{3} = -1.667 \qquad \csc\theta_2 = \tfrac{5}{4} = 1.250$$

EXAMPLE 3 Quadrant III reference angles

The trigonometric functions of $\theta_3 = 210°$ can be expressed in terms of the reference angle of 30° (see Fig. 9), and then evaluated, as follows:

same function; reference angle

$$\sin 210° = -\sin(210° - 180°) = -\sin 30° = -\frac{1}{2} = -0.5000$$

quadrant III; $\sin\theta$, $\csc\theta$ − in quadrant III

$$\csc 210° = -\csc 30° = -2.000$$

$$\cos 210° = -\cos 30° = -0.8660 \qquad \sec 210° = -\sec 30° = -1.155$$

$\cos\theta$, $\sec\theta$ − in quadrant III

$$\tan 210° = +\tan 30° = +0.5774 \qquad \cot 210° = +\cot 30° = +1.732$$

$\tan\theta$, $\cot\theta$ + in quadrant III

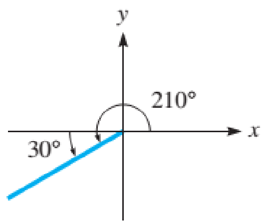


Fig. 9

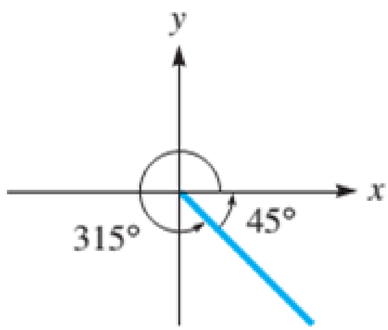


Fig. 10

EXAMPLE 4 Quadrant IV reference angles

If $\theta_4 = 315°$, the functions of θ_4 are found by using Eq. (5) as follows. See Fig. 10.

reference angle

$$\sin 315° = -\sin(360° - 315°) = -\sin 45° = -0.7071$$

quadrant IV — $\sin\theta$, $\csc\theta$ − in quadrant IV

$$\csc 315° = -\csc 45° = -1.414$$

$$\cos 315° = +\cos 45° = +0.7071 \qquad \sec 315° = +\sec 45° = +1.414$$

$\cos\theta$, $\sec\theta$ + in quadrant IV

$$\tan 315° = -\tan 45° = -1.000 \qquad \cot 315° = -\cot 45° = -1.000$$

$\tan\theta$, $\cot\theta$ − in quadrant IV

■

EXAMPLE 5 Evaluating using reference angles

same function — reference angle

$$\sin 160^\circ = +\sin(180^\circ - 160^\circ) = \sin 20^\circ = 0.3420$$

$$\tan 110^\circ = -\tan(180^\circ - 110^\circ) = -\tan 70^\circ = -2.747$$

$$\cos 225^\circ = -\cos(225^\circ - 180^\circ) = -\cos 45^\circ = -0.7071$$

$$\cot 260^\circ = +\cot(260^\circ - 180^\circ) = \cot 80^\circ = 0.1763$$

$$\sec 304^\circ = +\sec(360^\circ - 304^\circ) = \sec 56^\circ = 1.788$$

$$\sin 357^\circ = -\sin(360^\circ - 357^\circ) = -\sin 3^\circ = -0.0523$$

determines quadrant — proper sign for function in quadrant

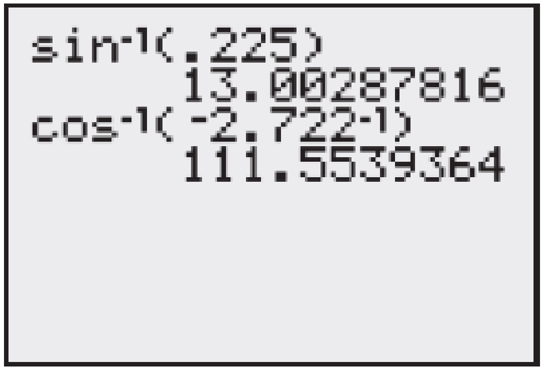


Fig. 13

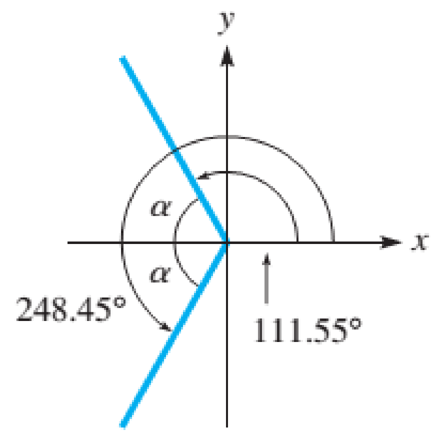


Fig. 15

EXAMPLE 8 Finding angles given $\sec\theta$

For $\sec\theta = -2.722$ and $0° \leq \theta < 360°$ (this means θ may equal 0° or be between 0° and 360°), see from the third and fourth lines of the calculator display shown in Fig. 13 that $\theta = 111.55°$ (rounded off).

The angle 111.55° is the second-quadrant angle, but $\sec\theta < 0$ in the third quadrant as well. The reference angle is $\alpha = 180° - 111.55° = 68.45°$, and the third-quadrant angle is $180° + 68.45° = 248.45°$. Therefore, the two angles between 0° and 360° for which $\sec\theta = -2.722$ are 111.55° and 248.45° (see Fig. 15). These angles can be checked by finding $\sec 111.55°$ and $\sec 248.45°$. ■

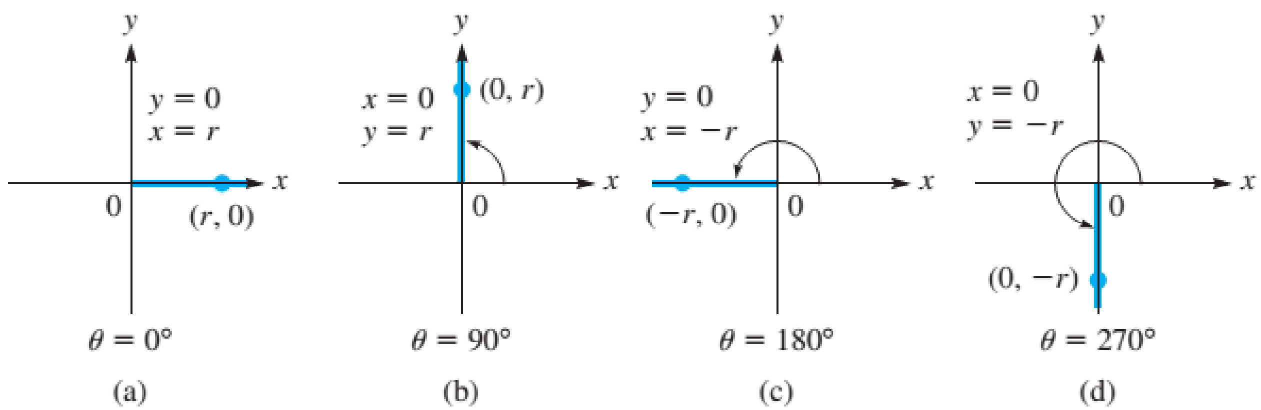


Fig. 18

EXAMPLE 11 Quadrantal angles

Because $\sin\theta = y/r$, by looking at Fig. 18(a), we can see that $\sin 0° = 0/r = 0$.

Because $\tan\theta = y/x$, from Fig. 18(b), we see that $\tan 90° = r/0$, which is undefined due to the division by zero. Using a calculator to find tan 90°, the display would indicate an error (due to division by zero).

Because $\cos\theta = x/r$, from Fig. 18(c), we see that $\cos 180° = -r/r = -1$.

Because $\cot\theta = x/y$, from Fig. 18(d), we see that $\cot 270° = 0/-r = 0$.

8.3 1,2,4, 7

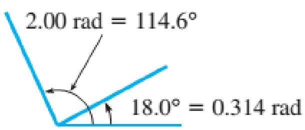


Fig. 25

EXAMPLE 1 Converting to and from radians

(a) converting degrees to radians; degrees cancel

$$18.0° = \left(\frac{\pi}{180°}\right)(18.0°) = \frac{\pi}{10.0} = 0.314 \text{ rad}$$

(See Fig. 25.)

(b) converting radians to degrees

$$2.00 \text{ rad} = \left(\frac{180°}{\pi}\right)(2.00) = \frac{360°}{\pi} = 114.6°$$

(See Fig. 25.)

Multiplying by $\pi/180°$ or $180°/\pi$ is actually multiplying by 1, because π rad = 180°. The unit of measurement is different, but *the angle is the same.* ■

Because of the definition of the radian, it is common to express radians in terms of π, particularly for angles whose degree measure is a fraction of 180°.

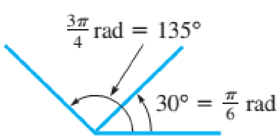


Fig. 26

EXAMPLE 2 Radians in terms of π

(a) Converting 30° to radian measure, we have

$$30° = \left(\frac{\pi}{180°}\right)(30°) = \frac{\pi}{6} \text{ rad}$$

(See Fig. 26)

(b) Converting $3\pi/4$ rad to degrees, we have

$$\frac{3\pi}{4} \text{ rad} = \left(\frac{180°}{\pi}\right)\left(\frac{3\pi}{4}\right) = 135°$$

(See Fig. 26) ■

EXAMPLE 4 Calculator evaluations

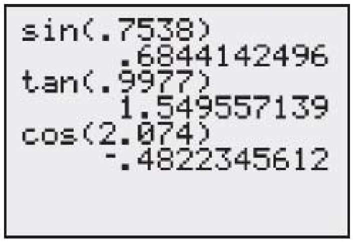


Fig. 29

no units indicates radian measure

(a) To find the value of sin 0.7538, put the calculator in radian mode (note that no units are shown with 0.7538), and the value is found as shown in the first two lines of the calculator display in Fig. 29. Therefore,

$$\sin 0.7538 = 0.6844$$

(b) From the third and fourth lines of the display in Fig. 29, we see that

$$\tan 0.9977 = 1.550$$

(c) From the last two lines of the display in Fig. 29, we see that

$$\cos 2.074 = -0.4822$$

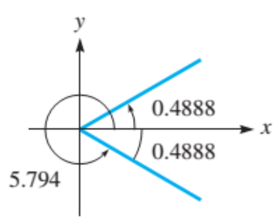
In each case, the calculator was in radian mode. ■

Practice Exercise

3. Find the value of sin 3.56.

EXAMPLE 7 Find angle in radians for given function

Express θ in radians, such that $\cos\theta = 0.8829$ and $0 \le \theta < 2\pi$.



Because $\cos\theta$ is positive and θ between 0 and 2π, we want a first-quadrant angle and a fourth-quadrant angle. With the calculator in radian mode (see Fig. 33),

$$\cos^{-1} 0.8829 = 0.4888 \quad \text{first-quadrant angle}$$

$$2\pi - 0.4888 = 5.794 \quad \text{fourth-quadrant angle}$$

$$\theta = 0.4888 \quad \text{or} \quad \theta = 5.794 \quad \text{see Fig. 34}$$ ■

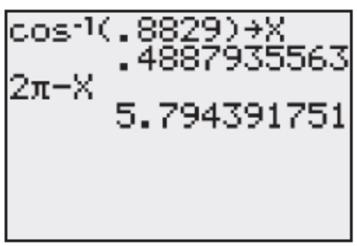


Fig. 34

8.4 1, 3,4,6

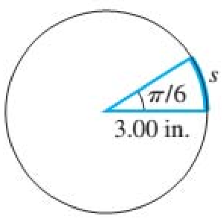


Fig. 38

EXAMPLE 1 Arc length

Find the length of arc on a circle of radius $r = 3.00$ in., for which the central angle $\theta = \pi/6$. See Fig. 38.

θ in radians

$$s = \left(\frac{\pi}{6}\right)(3.00) = \frac{\pi}{2.00}$$
$$= 1.57 \text{ in.}$$

Therefore, the length of arc s is 1.57 in. ■

AREA OF A SECTOR OF A CIRCLE

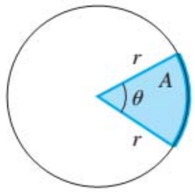


Fig. 40

Another application of radians is finding the area of a sector of a circle (see Fig. 40). Recall from geometry that areas of sectors of circles are proportional to their central angles. The area of a circle is $A = \pi r^2$, which can be written as $A = \frac{1}{2}(2\pi)r^2$. Because the angle for a complete circle is 2π, *the area of any sector of a circle in terms of the radius and central angle (in radians) is*

$$A = \frac{1}{2}\theta r^2 \qquad (\theta \text{ in radians}) \tag{12}$$

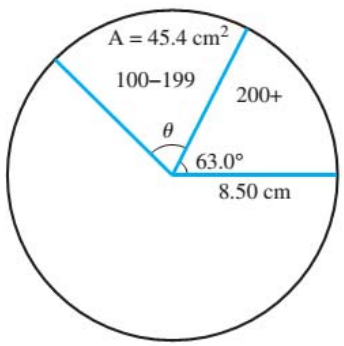


Fig. 41

EXAMPLE 3 Area of sector of circle—application

(a) Showing that 17.5% of high school students send at least 200 text messages per day on a pie chart of radius 8.50 cm means that the central angle of the sector is 63.0° (17.5% of 360°). The area of this sector (see Fig. 41) is

$$A = \frac{1}{2}(63.0)\left(\frac{\pi}{180}\right)(8.50^2) = 39.7 \text{ cm}^2 \qquad \theta \text{ in radians}$$

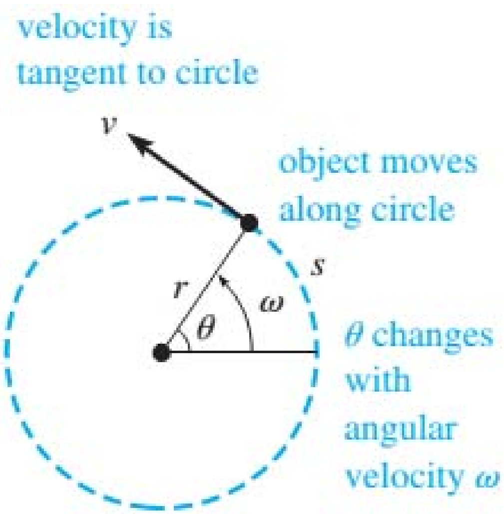
(b) Given that the area of the pie chart in Fig. 41 that shows the percent of students who send 100–199 text messages per day is 45.4 cm², we can find the central angle of this sector by first solving for θ. This gives

$$\theta = \frac{2A}{r^2} = \frac{2(45.4)}{8.50^2} = 1.26 \qquad \text{no units indicates radian measure}$$

This means the central angle is 1.26 rad, or 72.2°. ■

Practice Exercise

2. Find A if $r = 17.5$ in. and $\theta = 125°$.



$$\frac{s}{t} = \frac{\theta r}{t} = \frac{\theta}{t} r$$

where θ/t is called the *angular velocity* and is designated by ω. Therefore,

$$v = \omega r$$

EXAMPLE 4 Angular velocity—application

A person on a hang glider is moving in a horizontal circular arc of radius 90.0 m with an angular velocity of 0.125 rad/s. The person's linear velocity is

$$v = (0.125\,\text{rad/s})(90.0\,\text{m}) = 11.3\,\text{m/s}$$

(Remember that radians are numbers and are not included in the final set of units.) This means that the person is moving along the circumference of the arc at 11.3 m/s (40.7 km/h). ■

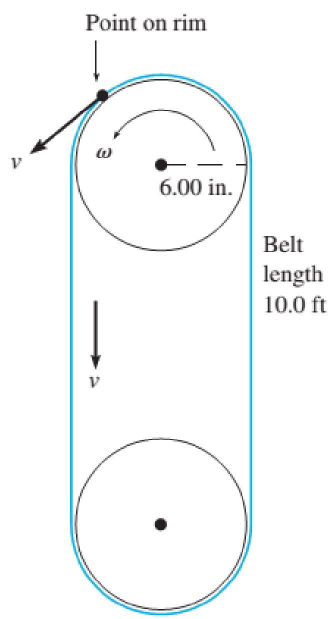


Fig. 43

EXAMPLE 6 Angular velocity—application

A pulley belt 10.0 ft long takes 2.00 s to make one complete revolution. The radius of the pulley is 6.00 in. What is the angular velocity (in revolutions per minute) of a point on the rim of the pulley? See Fig. 43.

Because the linear velocity of a point on the rim of the pulley is the same as the velocity of the belt, $v = 10.0/2.00 = 5.00\ \text{ft/s}$. The radius of the pulley is $r = 6.00\ \text{in.} = 0.500\ \text{ft}$, and we can find ω by substituting into Eq. (13). This gives us

$$
\begin{aligned}
v &= \omega r \\
5.00 &= \omega(0.500) \\
\omega &= 10.0\ \text{rad/s} && \text{multiply by } 60\ \text{s}/1\ \text{min} \\
&= 600\ \text{rad/min} && \text{multiply by } 1\ \text{r}/2\pi\ \text{rad} \\
&= 95.5\ \text{r/min} && \text{r is the symbol for revolution}
\end{aligned}
$$

The change of units can be handled algebraically as

$$10.0\,\frac{\text{rad}}{\text{s}} \times 60\,\frac{\text{s}}{\text{min}} = 600\,\frac{\text{rad}}{\text{min}}$$

$$\frac{600\ \text{rad/min}}{2\pi\ \text{rad/r}} = 600\,\frac{\text{rad}}{\text{min}} \times \frac{1}{2\pi}\,\frac{\text{r}}{\text{rad}} = 95.5\ \text{r/min}$$

Many types of problems use radian measure. Following is one involving electric current.

10.1 1,2,3

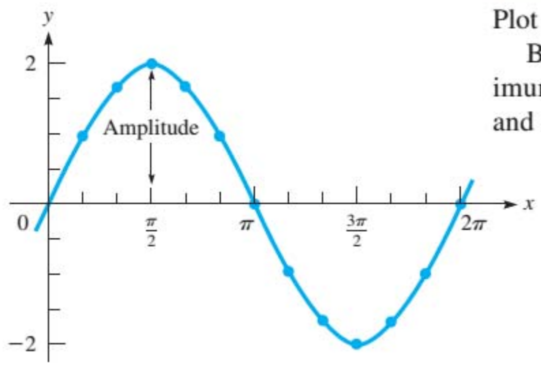


Fig. 5

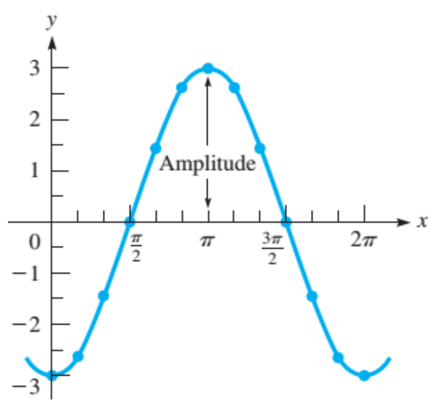
EXAMPLE 1 Plot graph of $y = a \sin x$

Plot the graph of $y = 2 \sin x$.

Because $a = 2$, the amplitude of this curve is $|2| = 2$. This means that the maximum value of y is 2 and the minimum value is $y = -2$. The table of values follows, and the curve is shown in Fig. 5.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	1	1.73	2	1.73	1	0

x	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	−1	−1.73	−2	−1.73	−1	0



EXAMPLE 2 Plot graph of $y = a \cos x$

Plot the graph of $y = -3 \cos x$.

In this case, $a = -3$, and this means that the amplitude is $|-3| = 3$. Therefore, the maximum value of y is 3, and the minimum value of y is −3. The table of values follows, and the curve is shown in Fig. 6.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	−3	−2.6	−1.5	0	1.5	2.6	3

x	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	2.6	1.5	0	−1.5	−2.6	−3

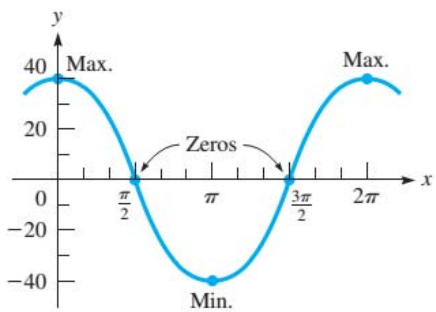


Fig. 7

EXAMPLE 3 Using key values to sketch graph

Sketch the graph of $y = 40 \cos x$.

First, we set up a table of values for the points where the curve has its zeros, maximum points, and minimum points:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	40 max.	0	−40 min.	0	40 max.

Now, we plot these points and join them, knowing the basic sinusoidal shape of the curve. See Fig. 7.

10.2 1,2,3

EXAMPLE 1 Finding period of a function

(a) The period of $\cos 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$.

(b) The period of $\sin 3\pi x$ is $\frac{2\pi}{3\pi} = \frac{2}{3}$.

(c) The period of $\sin \frac{1}{2}x$ is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

(d) The period of $\cos \frac{\pi}{4}x$ is $\dfrac{2\pi}{\frac{\pi}{4}} = 8$.

In (a), the period tells us that the curve of $y = \cos 4x$ will repeat every $\pi/2$ (approximately 1.57) units of x. In (b), we see that the curve of $y = \sin 3\pi x$ will repeat every 2/3 of a unit. In (c) and (d), the periods are longer than those of $y = \sin x$ and $y = \cos x$.

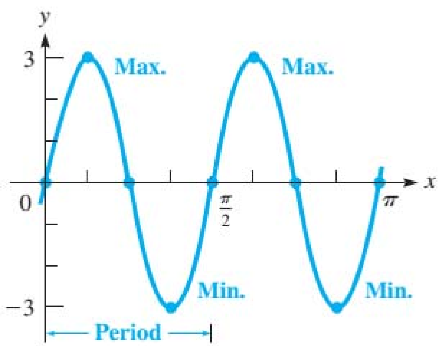


Fig. 10

EXAMPLE 2 Sketching graph of $y = a \sin bx$

Sketch the graph of $y = 3\sin 4x$ for $0 \leq x \leq 2\pi$.

Because $a = 3$, the amplitude is 3. The $4x$ tells us that the period is $2\pi/4 = \pi/2$. This means that $y = 0$ for $x = 0$ and for $y = \pi/2$. Because this sine function is zero halfway between $x = 0$ and $x = \pi/2$, we find that $y = 0$ for $x = \pi/4$. Also, the fact that the graph of the sine function reaches its maximum and minimum values halfway between zeros means that $y = 3$ for $x = \pi/8$, and $y = -3$ for $x = 3\pi/8$. Note that the values of x in the following table are those for which $4x = 0$, $\pi/2$, π, $3\pi/2$, 2π, and so on.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
y	0	3	0	- 3	0	3	0	- 3	0

Using the values from the table and the fact that the curve is sinusoidal in form, we sketch the graph of this function in Fig. 10. We see again that knowing the key values and the basic shape of the curve allows us to *sketch* the graph of the curve quickly and easily. ■

EXAMPLE 3 Using important values to sketch graph

Sketch the graph of $y = -2\cos 3x$ for $0 \le x \le 2\pi$.

Note that the amplitude is 2 and the period is $\frac{2\pi}{3}$. This means that one-fourth of the period is $\frac{1}{4} \times \frac{2\pi}{3} = \frac{\pi}{6}$. Because the cosine curve is at a maximum or minimum for $x = 0$, we find that $y = -2$ for $x = 0$ (the negative value is due to the minus sign before the function), which means it is a minimum point. The curve then has a zero at $x = \frac{\pi}{6}$, a maximum value of 2 at $x = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$, a zero at $x = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$, and its next value of -2 at $x = 4\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$, and so on. Therefore, we have the following table:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	-2	0	2	0	-2	0	2	0	-2	0	2	0	-2

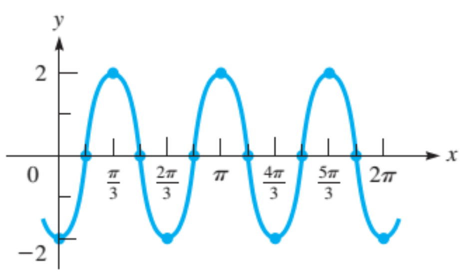


Fig. 11

Using this table and the sinusoidal shape of the cosine curve, we sketch the function in Fig. 11.

10.3 1,2,4

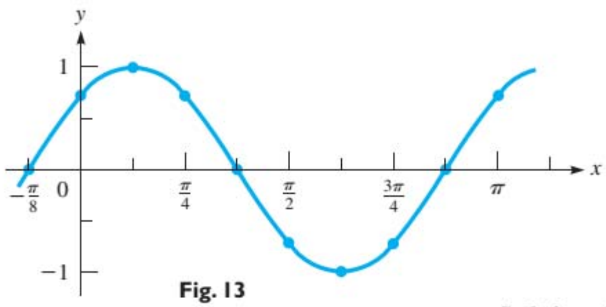


Fig. 13

EXAMPLE 1 Sketch function with phase angle

Sketch the graph of $y = \sin\left(2x + \frac{\pi}{4}\right)$.

Here, $c = \pi/4$. Therefore, in order to obtain values for the table, we assume a value for x, multiply it by 2, add $\pi/4$ to this value, and then find the sine of the result. The values shown are those for which $2x + \pi/4 = 0, \pi/4, \pi, 2, 3\pi/4, \pi$, and so on, which are the important values for $y = \sin 2x$.

x	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
y	0	0.7	1	0.7	0	−0.7	−1	−0.7	0	0.7

Solving $2x + \pi/4 = 0$, we get $x = -\pi/8$, which gives $y = \sin 0 = 0$. The other values for y are found in the same way. See Fig. 13. ■

EXAMPLE 2 Sketching graph of $y = a \sin(bx + c)$

Sketch the graph of $y = 2\sin(3x - \pi)$.

First, note that $a = 2, b = 3$, and $c = -\pi$. Therefore, the amplitude is 2, the period is $2\pi/3$, and the displacement is $-(-\pi/3) = \pi/3$. (We can also get the displacement from $3x - \pi = 0, x = \pi/3$.)

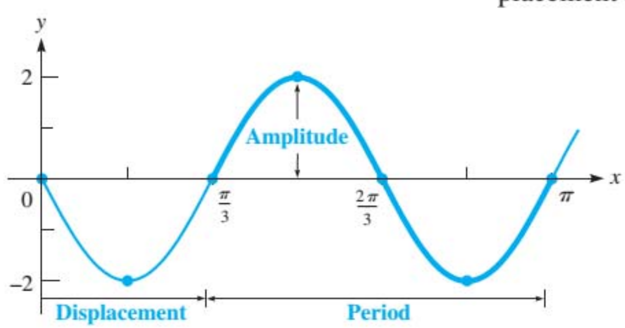


Fig. 16

Note that the curve "starts" at $x = \pi/3$ and starts repeating $2\pi/3$ units to the right of this point. Be sure to grasp this point well. *The period tells us the number of units along the x-axis between such corresponding points.* One-fourth of the period is $\frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$.

Important values are at $\frac{\pi}{3}, \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}, \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$, and so on. We now make the table of important values and sketch the graph shown in Fig. 16.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	−2	0	2	0	−2	0

$\leftarrow \frac{1}{4}(\text{period}) = \frac{\pi}{6}$

Note that because the period is $2\pi/3$, the curve passes through the origin. ■

EXAMPLE 3 Sketching graph of $y = a\cos(bx + c)$

Sketch the graph of the function $y = -\cos\left(2x + \frac{\pi}{6}\right)$.

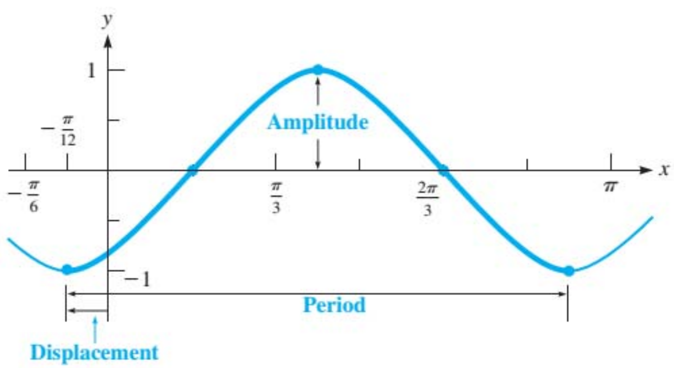


Fig. 17

First, we determine that

1. the amplitude is 1
2. the period is $\frac{2\pi}{2} = \pi$
3. the displacement is $-\frac{\pi}{6} \div 2 = -\frac{\pi}{12}$

We now make a table of important values, noting that the curve starts repeating π units to the right of $-\frac{\pi}{12}$.

x	$-\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{5\pi}{12}$	$\frac{2\pi}{3}$	$\frac{11\pi}{12}$
y	−1	0	1	0	−1

← $\frac{1}{4}$(period) = $\frac{\pi}{4}$

From this table, we sketch the graph in Fig. 17.

EXAMPLE 4 Graph on calculator

View the graph of $y = 2\cos\left(\frac{1}{2}x - \frac{\pi}{6}\right)$ on a calculator.

From the values $a = 2, b = 1/2$, and $c = -\pi/6$, we determine that

1. the amplitude is 2
2. the period is $2\pi \div \frac{1}{2} = 4\pi$
3. the displacement is $-\left(-\frac{\pi}{6}\right) \div \frac{1}{2} = \frac{\pi}{3}$

We now make a table of important values:

x	$\frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{7\pi}{3}$	$\frac{10\pi}{3}$	$\frac{13\pi}{3}$
y	2	0	−2	0	2

← $\frac{1}{4}$(period) $= \pi$

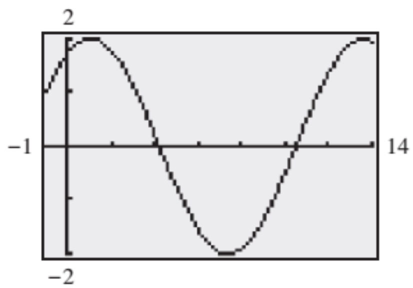


Fig. 18

This table helps us choose the values for the *window* settings in Fig. 18. We choose Xmin = −1 in order to start to the left of the y-axis and Xmax = 14 because $13\pi/3 \approx 13.6$. Also, we choose Ymin = −2 and Ymax = 2 because the amplitude is 2. We see that the graph in Fig. 18 is a little more than one cycle.

