Mathematics 2 (Maths 102)

Dr Peter Lawson

Course Topics

- Trigonometric functions.
- Vectors.
- Graphs of the trigonometric functions
- Solving systems of two linear equations.
- Determinants.
- Statistics.

Assessments

Weekly quizzes, weekly problem sets, midterm examination, and final examination.

<u>Grading policy:</u>

Weekly Quizzes	(20%)
Problem Sets	(20%)
Midterm Exam	(25%)
Final Exam	(35%)
Total	(100%)

Chapter 4

The Trigonometric Functions

Learning Outcomes

- At the end of this chapter the student will:
  - Describe how angles are defined.
  - Describe how the trigonometric functions are defined.
  - Determine the values for the trigonometric functions for angles.
  - Determine the angle when a trigonometric ratio is given.
  - Apply concepts related to the trigonometric ratios.

1) Compute the unknown angle of right and oblique triangles applying law of sine and cosine

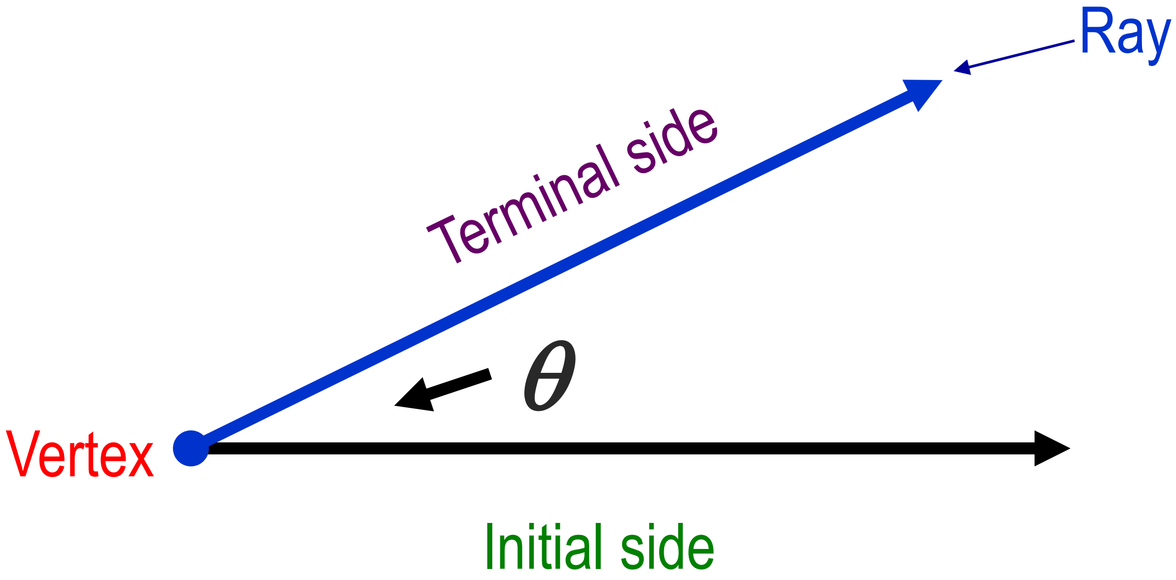
- 1.1 Given a right triangle and a reference angle, IDENTIFY the following triangle sides: Adjacent, Opposite, Hypotenuse.
- 1.2 From memory, STATE the formulas for the following trigonometric functions: Sine, Cosine, Tangent, Cosecant, Secant, and Cotangent.
- 1.3 Given a right triangle and a reference angle, the length of any side, and with an approved calculator; SOLVE for the unknown angles of the triangle.
- 1.4 Given a right triangle and a reference angle, the length of any side, and with an approved calculator, SOLVE for the unknown angles and sides of the triangle.
- 1.5 Given a triangle other than a right angle triangle, a reference angle, the length of any side, and with an approved calculator, SOLVE for the unknown angles and sides of the triangle.
- 1.6 Given a right triangle and the length of any two sides, SOLVE for the length of the third side using the Pythagorean Theorem.

Ch. 4.1: Angles

- An ***angle*** is generated by rotating a ***ray*** about its fixed endpoint from an initial position to a terminal position.
- The initial position is called the ***initial side*** of the angle, the terminal position is called the ***terminal side***, and the fixed endpoint is the ***vertex***.

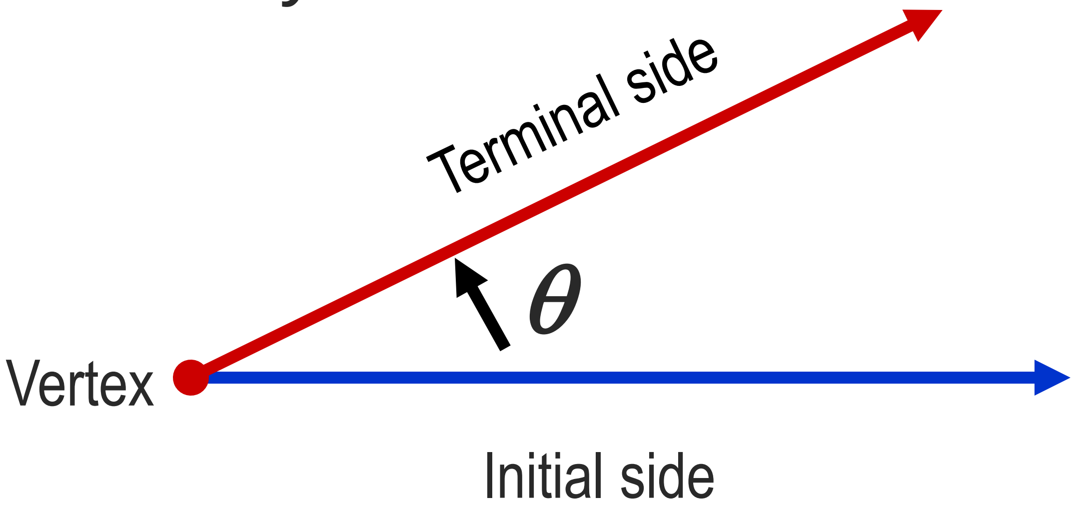
Angles

- Defining an angle:



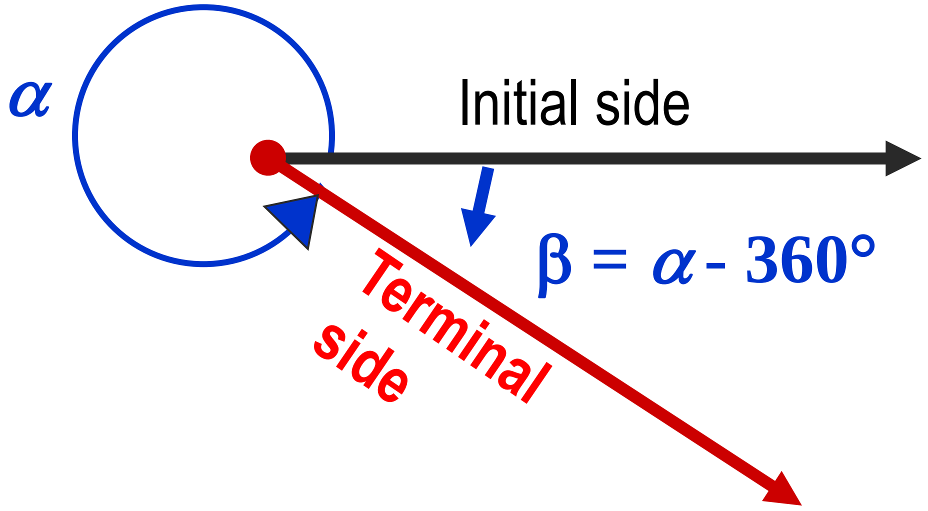
Angles Formed by Rotation

- Positive angles: counterclockwise rotation
- Denoted by Greek letters.



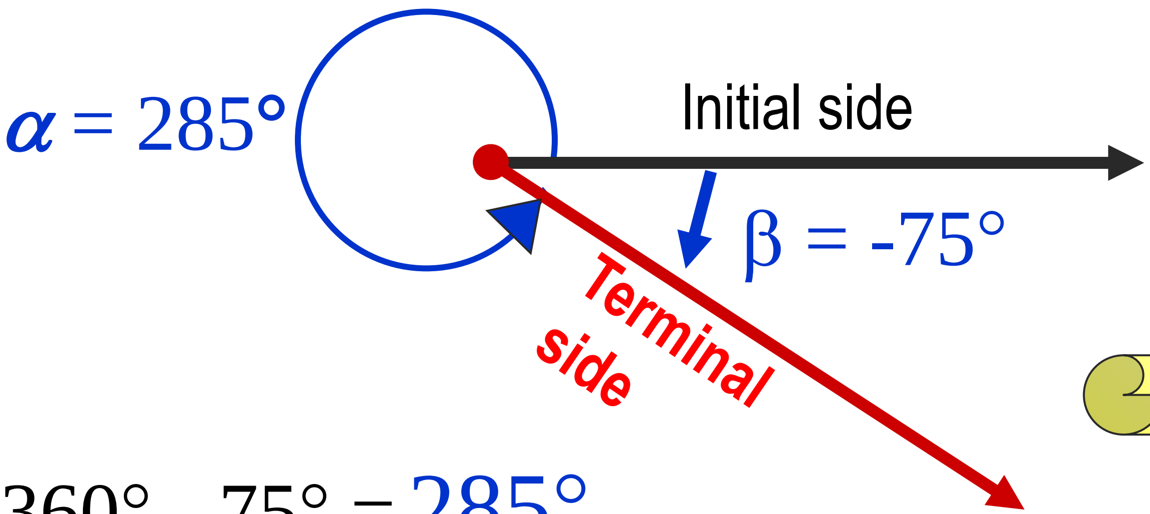
Angles Formed by Rotation

- Positive angles can also be written as ***negative*** angles: clockwise rotation
- Also, denoted by Greek letters.



Angles Formed by Rotation

- Write -75° as a positive angle.



> The angles, α and β, are called coterminal angles since they share the same initial & terminal sides.

$$360° - 75° = 285°$$

Angle Conversions

- $360° = 2\pi$ radians = 1 revolution
- 1 degree = 60 minutes
- 1 minute = 60 seconds

There are 3600 sec in 1 deg.

Angle Conversions > Min to Degrees

- **Example 1**:
- Convert 65°25’ to decimal degree form.

$$65°25' = 65° + \frac{25}{60} = 65.42°$$

Divide by 60.

Angle Conversions > Degrees to minutes

- **Example #2**
- Convert 32.459° to minute form.

$$32.459^\circ$$
$$= 32^\circ + 0.459 \times 60$$
$$= 32^\circ + 26.34'$$

Multiply by 60.

Convert 32.459° to degree/minute/second form.

$$32.459°$$
$$= 32° + 0.459 \times 60$$
$$= 32° + 26.34'$$
$$= 32°26' + 0.34 \times 60$$
$$= 32°26'20.4''$$

Angle Conversions

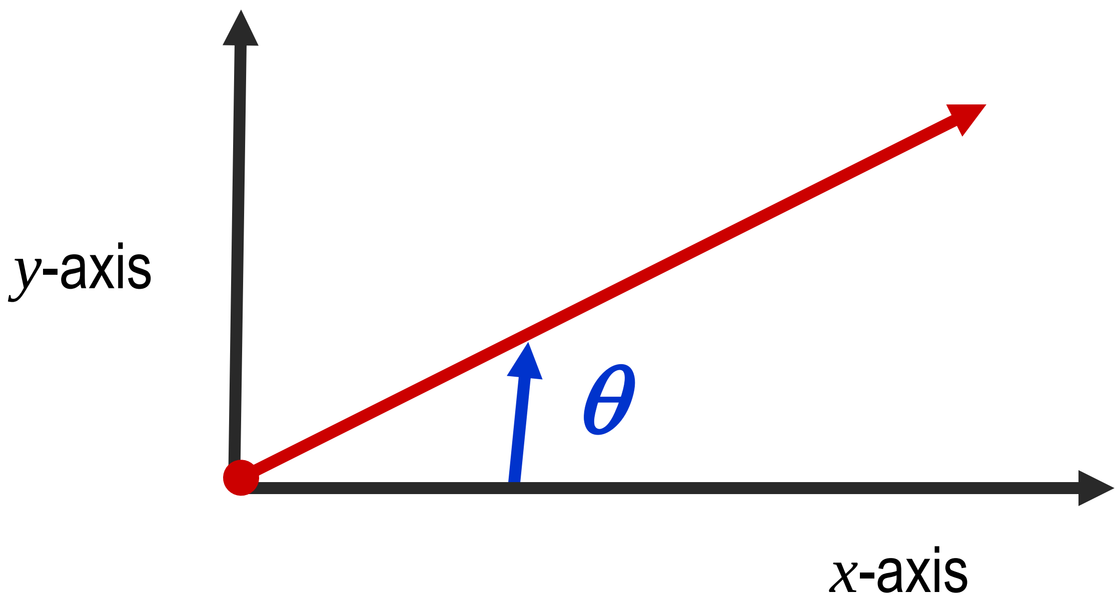
- There are **360° in 2π rads** in a full circle.

- Multiply angle in degrees to find rads by: $\frac{\pi}{180}$

- Multiply angle in rads to find degrees by: $\frac{180}{\pi}$

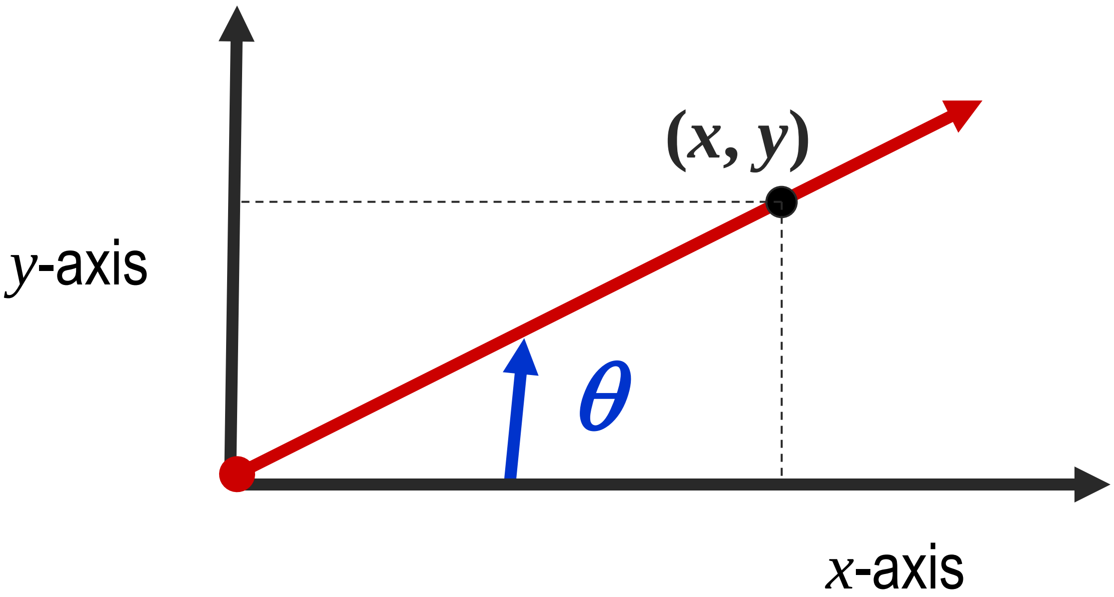
Standard Position of an Angle

- If the initial side of the angle is the positive ***x***-axis, and the vertex is the origin, the angle is said to be in ***standard position***.



Standard Position of an Angle

- The terminal side of an angle is uniquely determined by knowing that it passes through the point **(*x*, *y*)**.



? Ex 4.1 q 9-15 & 17-20

In Exercises 9–16, determine one positive and one negative coterminal angle for each angle given.

9. 45° **10.** 73°

11. −150° **12.** 462°

13. 70°30′ **14.** 153°47′

15. 278.1° **16.** −197.6°

In Exercises 17–20, by means of the definition of a radian, change the given angles in radians to equal angles expressed in degrees to the nearest 0.01°.

17. 0.265 rad **18.** 0.838 rad

19. 1.447 rad **20.** −3.642 rad

17. To change 0.265 rad to degrees multiply by $\frac{180}{\pi}$,

$$0.265\text{ rad}\left(\frac{180^\circ}{\pi\text{ rad}}\right) \approx 15.18^\circ$$

9. positive: $45^\circ + 360^\circ = 405^\circ$

negative: $45^\circ - 360^\circ = -315^\circ$

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9. positive: $45^\circ + 360^\circ = 405^\circ$

negative: $45^\circ - 360^\circ = -315^\circ$

10. $73^\circ + 360^\circ = 433^\circ$

$73^\circ - 360^\circ = -287^\circ$

11. $-150^\circ + 360^\circ = 210^\circ$

$-150^\circ - 360^\circ = -510^\circ$

12. $462^\circ + 360^\circ = 822^\circ$

$462^\circ - 2(360^\circ) = -258^\circ$

13. positive: $70^\circ 30' + 360 = 430^\circ 30'$

negative: $70^\circ 30' - 360^\circ = -289^\circ 30'$

14. $153^\circ 47' + 360^\circ = 513^\circ 47'$

$153^\circ 47' = -360^\circ = -206^\circ 13'$

15. $278.1^\circ + 360^\circ = 638.1^\circ$

$278.1^\circ - 360^\circ = -81.9^\circ$

17. To change 0.265 rad to degrees multiply by $\frac{180}{\pi}$,

$0.265 \text{ rad} \left(\frac{180^\circ}{\pi \text{ rad}}\right) \approx 15.18^\circ$

18. To change 0.838 rad to degrees multiply by $180^\circ / \pi$, which gives 48.01; 0.0838 rad $= 82.91^\circ$

19. To change 1.447 rad to degrees multiply by $180^\circ / \pi$, which gives 82.91; 1.447 rad $= 82.91^\circ$

20. To change -3.642 rad to degrees multiply by $180^\circ / \pi$, which gives -208.67; -3.642 rad $= -208.67^\circ$

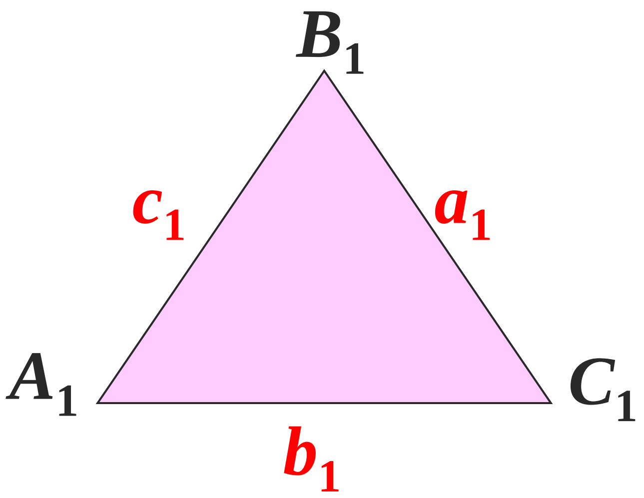
Ch. 4.2: Defining the Trigonometric Functions

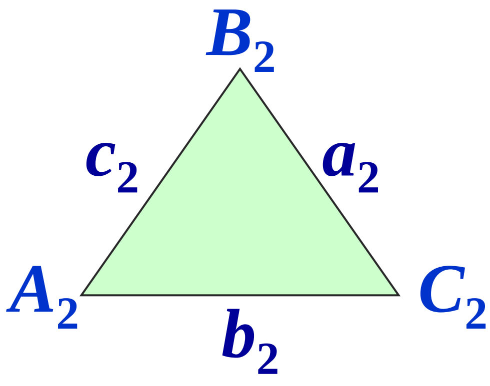
- The value of using trigonometric functions lies in an understanding of properties of similar triangles.
- ***Properties of Similar Triangles***:

1. Corresponding angles are equal.
2. Corresponding sides are proportional.

Properties of Similar Triangles

- The corresponding sides are the sides, one in each triangle, that are between the same pair of equal corresponding angles.





Properties of Similar Triangles

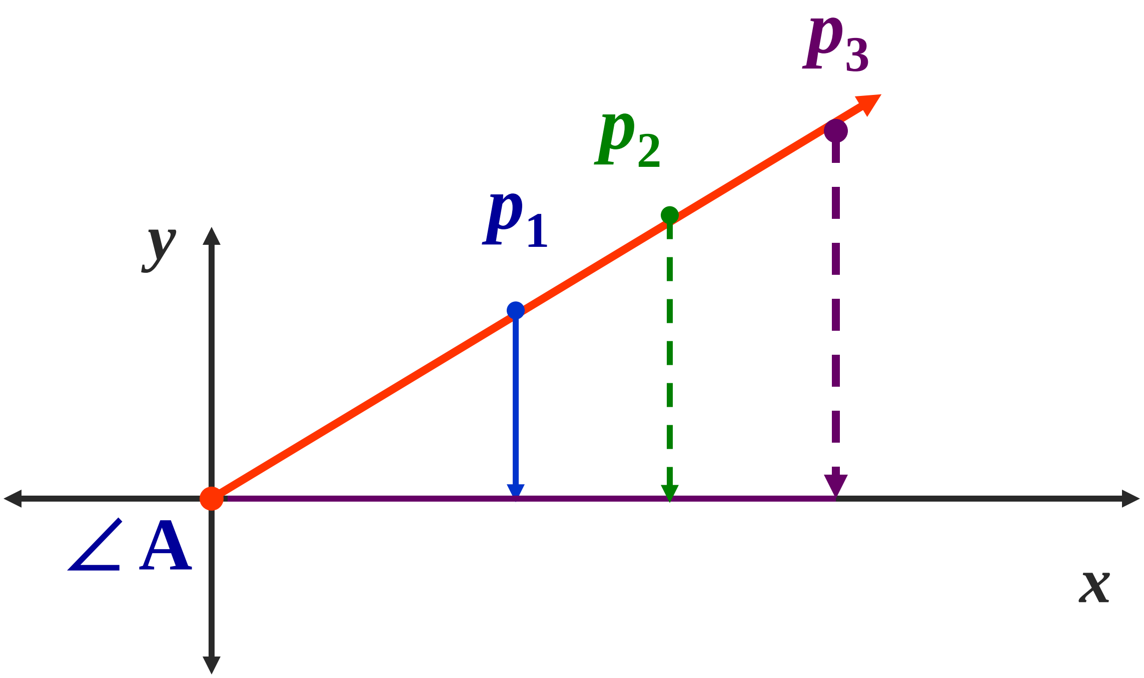
- From this definition of corresponding angles, we can determine that:

the ratio of one side to another side in one triangle is the same as the ratio of the corresponding sides in the other triangle.

- We apply this concept to our understanding of trigonometry.

Determining the Trigonometric Ratios

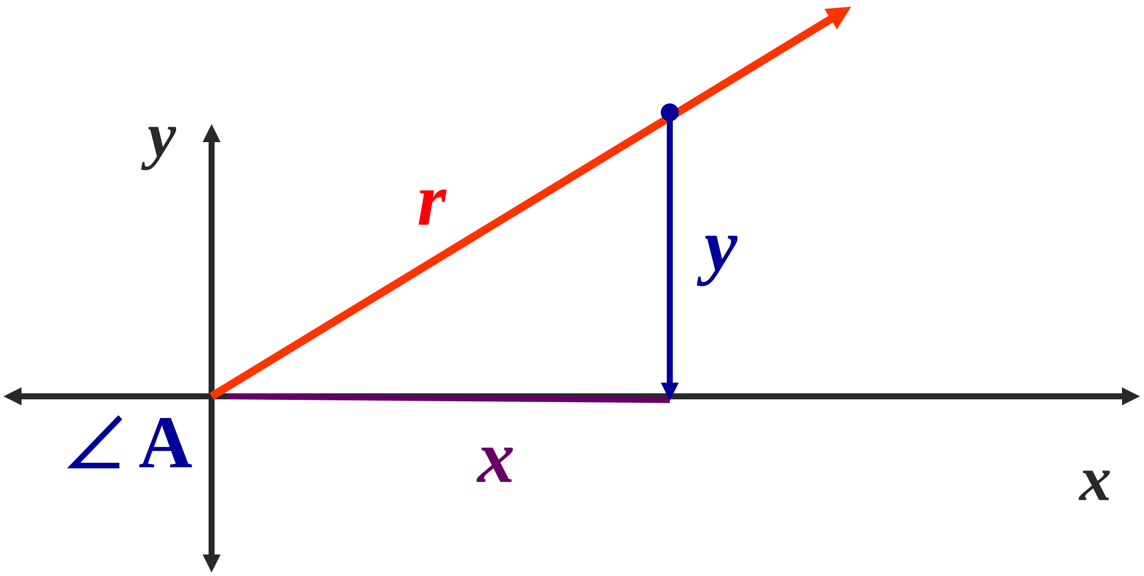
Regardless how far ***p*** is away from ∠ A, the size of the angle will never change.



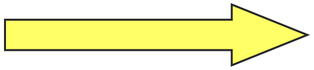
Therefore, the ***ratio*** between the lengths of the sides will never change.

Determining the Trigonometric Ratios

We label the right triangle as:

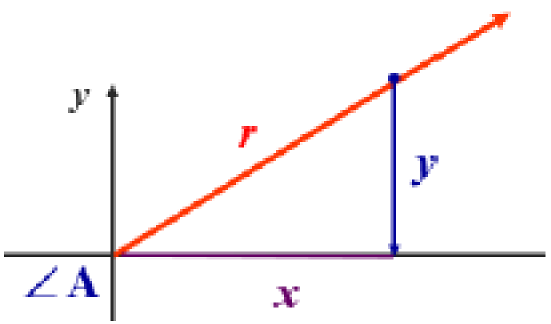


The trigonometric ratios are defined as follows:



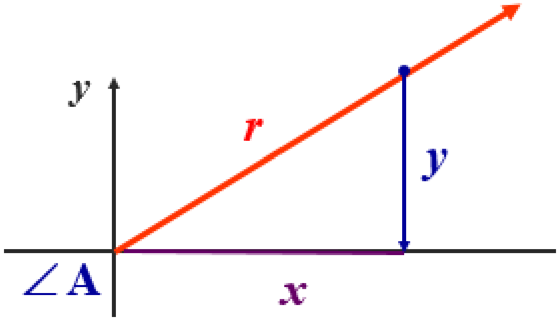
The Trigonometric Functions

- Sine of θ: $\sin\theta = \dfrac{y}{r}$
- Cosecant of θ: $\csc\theta = \dfrac{r}{y}$
- Cosine of θ: $\cos\theta = \dfrac{x}{r}$
- Secant of θ: $\sec\theta = \dfrac{r}{x}$
- Tangent of θ: $\tan\theta = \dfrac{y}{x}$
- Cotangent of θ: $\cot\theta = \dfrac{x}{y}$



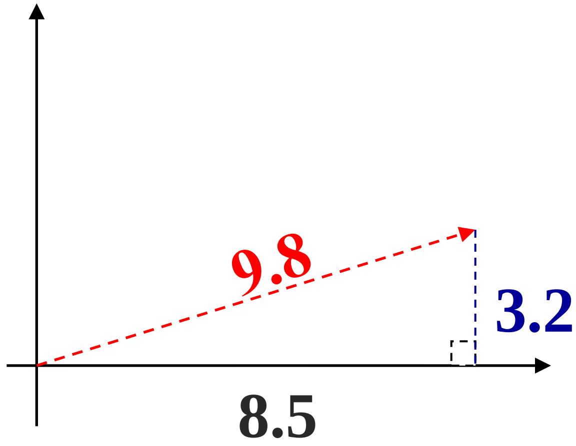
Evaluating the Trigonometric Functions

- The values of a trigonometric function are dependent on:
  - The ratios of the sides
  - The Pythagorean Theorem



Example

- Find the values of the trigonometric functions of the angle whose terminal side passes through **(8.5, 3.2)**.



sin A = 3.2/9.08 = 0.352

csc A = 1/0.352 = 2.838

cos A = 8.5/9.08 = 0.936

sec A = 1/0.936 = 1.068

tan A = 3.2/8.5 = 0.376

cot A = 1/0.376 = 2.656

csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A

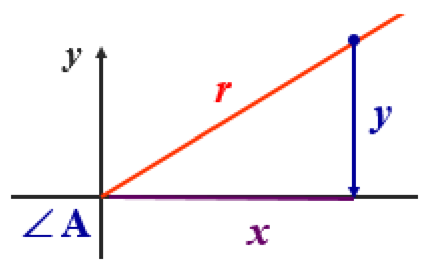
Ex 4.2 q 17-19 & 29-32

17. $\cos\theta = \frac{12}{13} \Rightarrow x = 12$ and $r = 13$ with θ in QL

$r^2 = x^2 + y^2 \Rightarrow 169 = 144 + y^2 \Rightarrow y^2 = 25$

$y = 5$

$\sin\theta = \frac{y}{r} = \frac{5}{13}$, $\cot\theta = \frac{x}{y} = \frac{12}{5}$.



29. $\sin^2\theta + \cos^2\theta = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} = \frac{16}{25}$

$= \frac{25}{25} = 1$

In Exercises 17–24, find the values of the indicated functions. In Exercises 17–20, give answers in exact form. In Exercises 21–24, the values are approximate.

17. Given $\cos\theta = 12/13$, find $\sin\theta$ and $\cot\theta$.

18. Given $\sin\theta = 1/2$, find $\cos\theta$ and $\csc\theta$.

19. Given $\tan\theta = 2$, find $\sin\theta$ and $\sec\theta$.

In Exercises 29–36, answer the given questions.

29. If $\tan\theta = 3/4$, what is the value of $\sin^2\theta + \cos^2\theta$? $[\sin^2\theta = (\sin\theta)^2]$

30. If $\sin\theta = 2/3$, what is the value of $\sec^2\theta - \tan^2\theta$?

31. If $y = \sin\theta$, what is $\cos\theta$ in terms of y?

32. If $x = \cos\theta$, what is $\tan\theta$ in terms of x?

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17. $\cos\theta = \frac{12}{13} \Rightarrow x = 12$ and $r = 13$ with θ in QI.

$r^2 = x^2 + y^2 \Rightarrow 169 = 144 + y^2 \Rightarrow y^2 = 25$

$y = 5$

$\sin\theta = \frac{y}{r} = \frac{5}{13}$, $\cot\theta = \frac{x}{y} = \frac{12}{5}$.

18. $\sin\theta = \frac{1}{2} = \frac{y}{r}$

$r^2 = x^2 + y^2;\ 2^2 = x^2 + 1;\ x^2 = 3;\ x = \pm\sqrt{3}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$ for acute θ

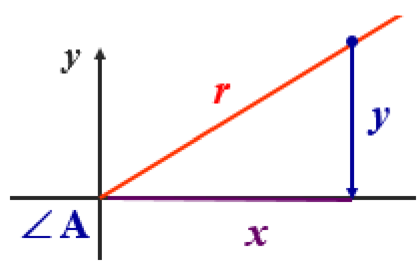
$\csc\theta = \frac{r}{y} = \frac{2}{1} = 2$ for acute θ

19. $\tan\theta = 2 = \frac{y}{x}$

$r^2 = x^2 + y^2 = 1^2 + 2^2 = 5;\ r = \pm\sqrt{5}$

$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{5}}$ for acute θ

$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{1} = \sqrt{5}$ for acute θ



29. $\sin^2\theta + \cos^2\theta = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} = \frac{16}{25}$

$= \frac{25}{25} = 1$

30. $\sin\theta = \frac{y}{r} = \frac{2}{3} \Rightarrow y = 2,\ r = 3$

$x = \sqrt{r^2 - y^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$

$\sec\theta = \frac{r}{x} = \frac{3}{\sqrt{5}}$, $\tan\theta = \frac{y}{x} = \frac{2}{\sqrt{5}}$

$\sec^2 - \tan^2\theta = \left(\frac{3}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = 1$

31. $\sin\theta = \frac{y}{r} = y \Rightarrow r = 1$

$\cos\theta = \frac{x}{r} = x = \sqrt{r^2 - y^2} = \sqrt{1 - \sin^2\theta}$

32. $\cos\theta = \frac{x}{r} = x \Rightarrow r = 1$

$\tan\theta = \frac{y}{x} = \sqrt{\frac{r^2 - x^2}{x}} = \sqrt{\frac{1 - x^2}{x}}$

Ch. 4.3: Values of the Trigonometric Functions

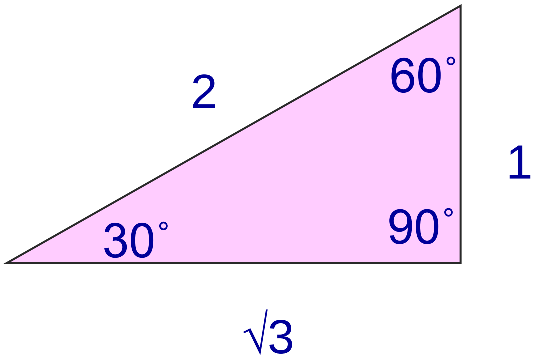
- If given a point on the plane, we can determine the trigonometric ratios of the angle made by the terminal side defined by that same point.
- There are some angles whose trigonometric values you should be familiar with.

Ch. 4.3: Values of the Trigonometric Functions

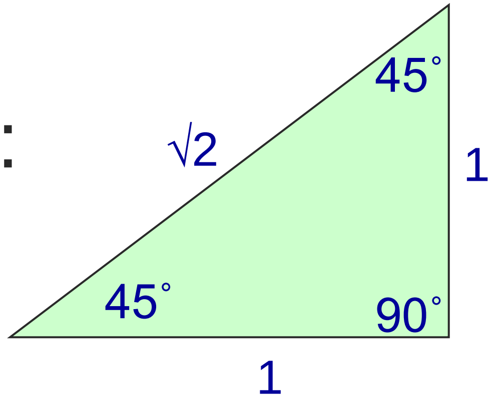
- Using our understanding of the 30°-60°-90° triangle and the 45°-45°-90° triangle, trigonometric ratios can be readily determined.
- Scale drawings of these triangles are used to calculate these ratios.
- It is helpful to be familiar with these values as they commonly occur.

Values of the Trigonometric Functions

- The 30°-60°-90° triangle:



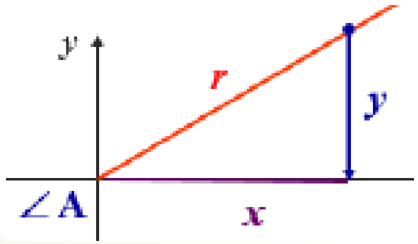
- The 45°-45°-90° triangle:



TASK

Ratios (sin,cos,tan) ?

Finding Unknown Angles



- We determine the unknown angles using the ***inverse*** trigonometric keys of the calculator.
- The notation used for the ratios is:

$$\theta = \sin^{-1}\frac{y}{r} \qquad \theta = \cos^{-1}\frac{x}{r} \qquad \theta = \tan^{-1}\frac{y}{x}$$

- Another commonly used notation is ***arcsin***, ***arccos*** and ***arctan***.

Finding Unknown Angles

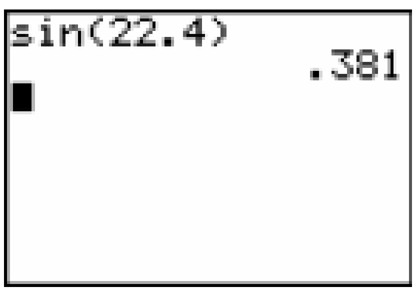
- If **sin *A* = 0.496** then ***A* = sin^{-1} 0.496**
- *Before you begin, do you want the angle in **degrees**? If so, make sure your calculator is on **degree** mode!*
- ***A* = sin^{-1} 0.496 = 29.74°**

• To change the settings for the number of decimal places, the number of significant digits, or the exponential display format, press the MODE key a number of times until you reach the setup screen shown below.

Fix	Sci	Norm
1	2	3

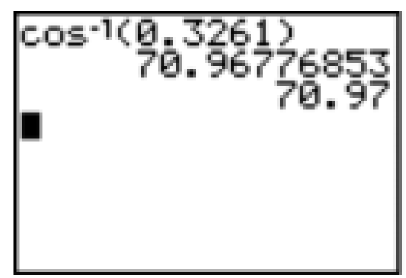
Ex 4.3 q 9-12 & 25-28

9. $\sin 22.4^\circ = 0.381$



25. $\cos\theta = 0.3261$

$\theta = 70.97^\circ$



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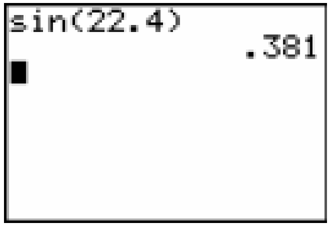
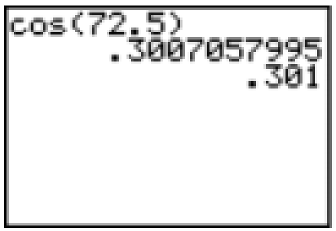
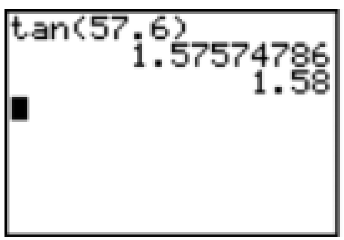
9. $\sin 22.4^\circ = 0.381$



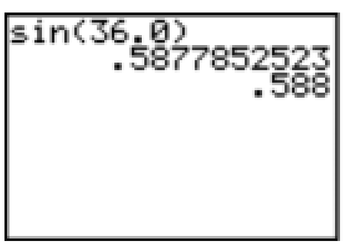
10. $\cos 72.5^\circ = 0.301$



11. $\tan 57.6^\circ = 1.58$

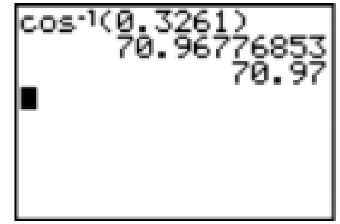


12. $\sin 36.0^\circ = 0.588$



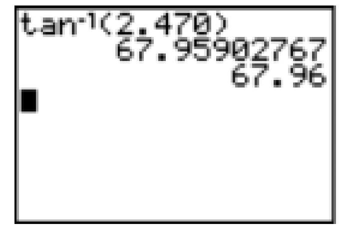
25. $\cos\theta = 0.3261$

$\theta = 70.97^\circ$



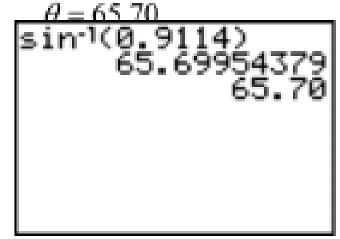
26. $\tan\theta = 2.470$

$\theta = 67.97^\circ$



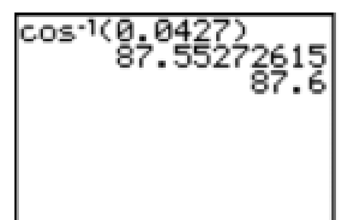
27. $\sin\theta = 0.9114$

$\theta = 65.70$



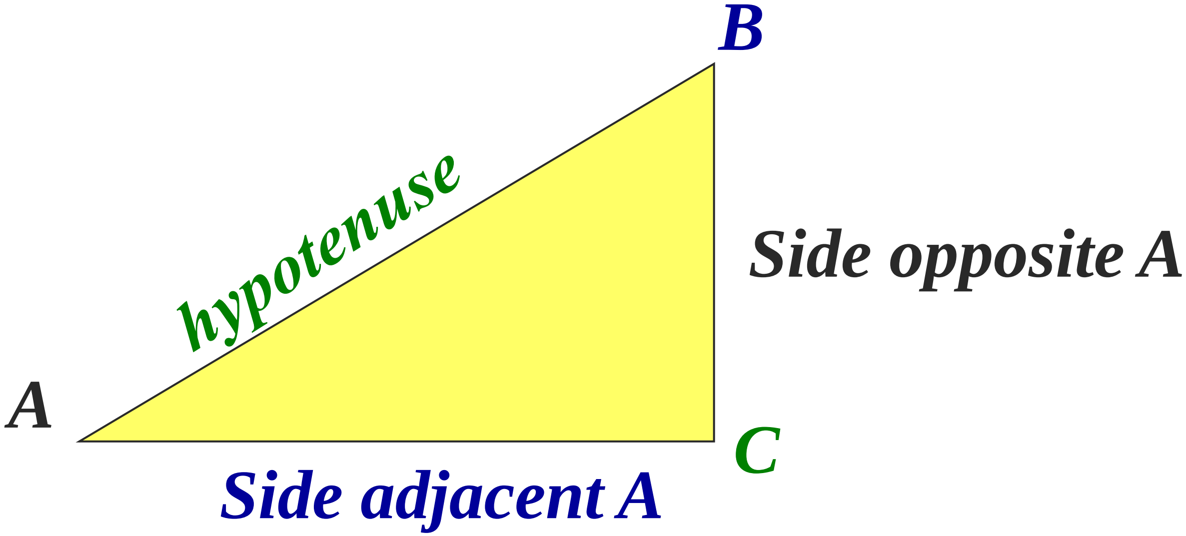
28. $\cos\theta = 0.0427$

$\theta = 87.6^\circ$



Ch. 4.4: The Right Triangle

- We can generalize the definitions of the trigonometric functions by naming the sides of a right angle triangle.



The Trigonometric Functions

- Sine of θ: $$\sin\theta = \frac{\text{side opposite } A}{\text{hypotenuse}}$$
- Cosine of θ: $$\cos\theta = \frac{\text{side adjacent } A}{\text{hypotenuse}}$$
- Tangent of θ: $$\tan\theta = \frac{\text{side opposite } A}{\text{side adjacent } A}$$

SOH CAH TOA

The Reciprocal Functions

- Cosecant of θ: $\csc\theta = \dfrac{hypotenuse}{side\ opposite\ A}$

- Secant of θ: $\sec\theta = \dfrac{hypotenuse}{side\ adjacent\ A}$

- Cotangent of θ: $\cot\theta = \dfrac{side\ adjacent\ A}{side\ opposite\ A}$

Procedure for Solving a Right Triangle

1. Sketch a right triangle and label the known and unknown sides and angles.
2. Express each of the 3 unknown parts in terms of the known parts and solve for the unknown parts.
3. Check the results.

Procedure for Solving a Right Triangle

- The sum of the angles should be 180°.
- If only one side is given, check the computed side with the ***Pythagorean Theorem***.
- If 2 sides are given, check the angles and computed side by using appropriate trigonometric functions.

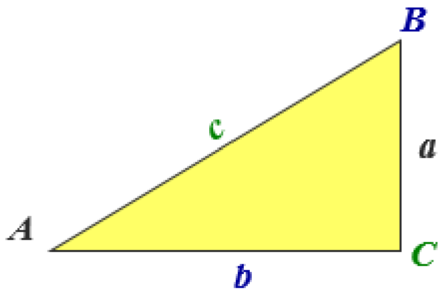
Ex 4.4 q 9-11 & 33-36

9. $\sin 77.8^\circ = \frac{6700}{c} \Rightarrow c = 6850$

$\angle B = 90^\circ - 77.8^\circ = 12.2^\circ$

$\tan 77.8^\circ = \frac{6700}{b} \Rightarrow b = 1450$

33. $\sin 61.7^\circ = \frac{3.92}{x} \Rightarrow x = \frac{3.92}{\sin 61.7^\circ} = 4.45$



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9. $\sin 77.8^\circ = \frac{6700}{c} \Rightarrow c = 6850$

$\angle B = 90^\circ - 77.8^\circ = 12.2^\circ$

$\tan 77.8^\circ = \frac{6700}{b} \Rightarrow b = 1450$

10. $B = 90.0^\circ - 18.4^\circ = 71.6^\circ$

$\sin A = \frac{a}{c}$

$a = c \sin A = 0.0897(\sin 18.4^\circ) = 0.0283$

$\cos A = \frac{b}{c}$

$b = c(\cos A) = 0.0897(\cos 18.4^\circ) = 0.0851$

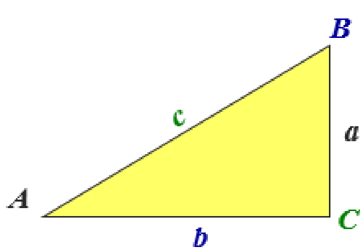
11. $b = \sqrt{345^2 - 150^2} = 311$

$\sin A = \frac{a}{c} = \frac{150}{345}$

$A = \sin^{-1}\left(\frac{150}{345}\right) = 25.8^\circ$

$\cos B = \frac{a}{c} = \frac{150}{345} = 0.435$

$B = \cos^{-1}\left(\frac{150}{345}\right) = 64.2^\circ$



33. $\sin 61.7^\circ = \frac{3.92}{x} \Rightarrow x = \frac{3.92}{\sin 61.7^\circ} = 4.45$

34. $\tan A = \frac{19.7}{36.3}$

$A = \tan^{-1}\left(\frac{19.7}{36.3}\right) = 28.5^\circ$

35. $\cos A = \frac{0.6673}{0.8742}$

$A = \cos^{-1}\left(\frac{0.6673}{0.8742}\right) = 40.24^\circ$

36. $\sin 22.45^\circ = \frac{x}{7265}$

$x = 7265 \sin 22.45^\circ$

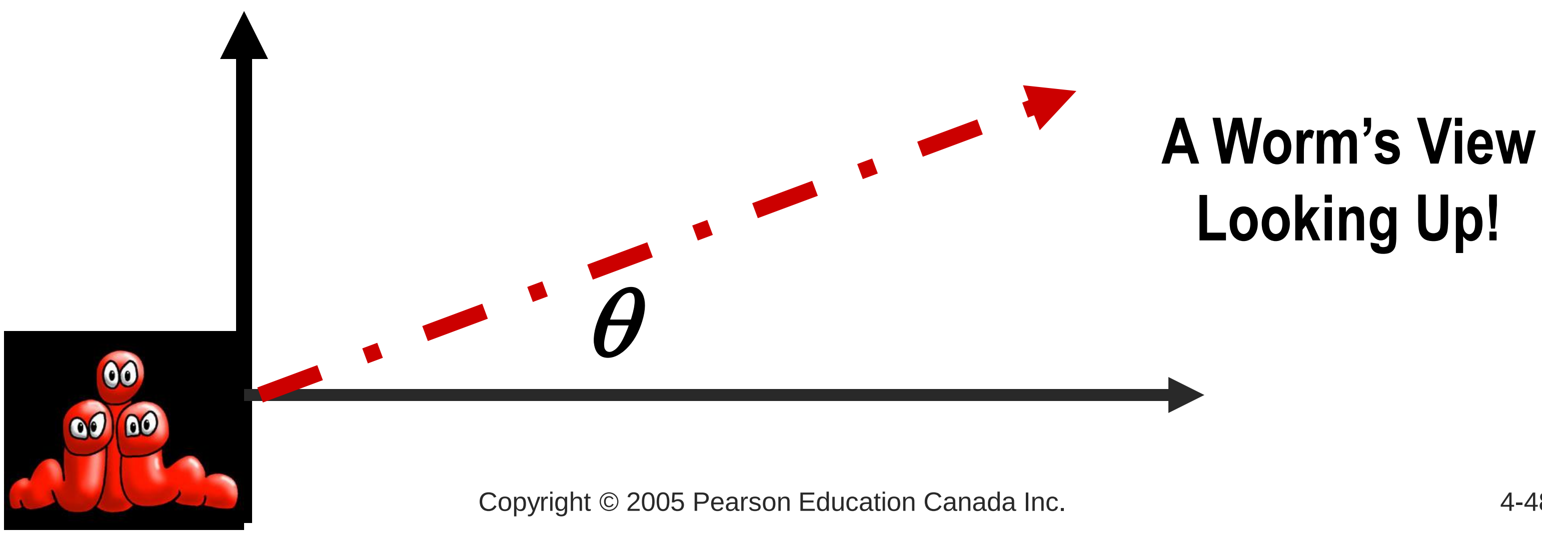
$= 2774$

Ch. 4.5: Applications of Right Triangles

- Situations where a right angle exists are easily solved using trigonometry.
- Here we introduce the ideas of:
  - the angle of depression, and,
  - the angle of elevation.

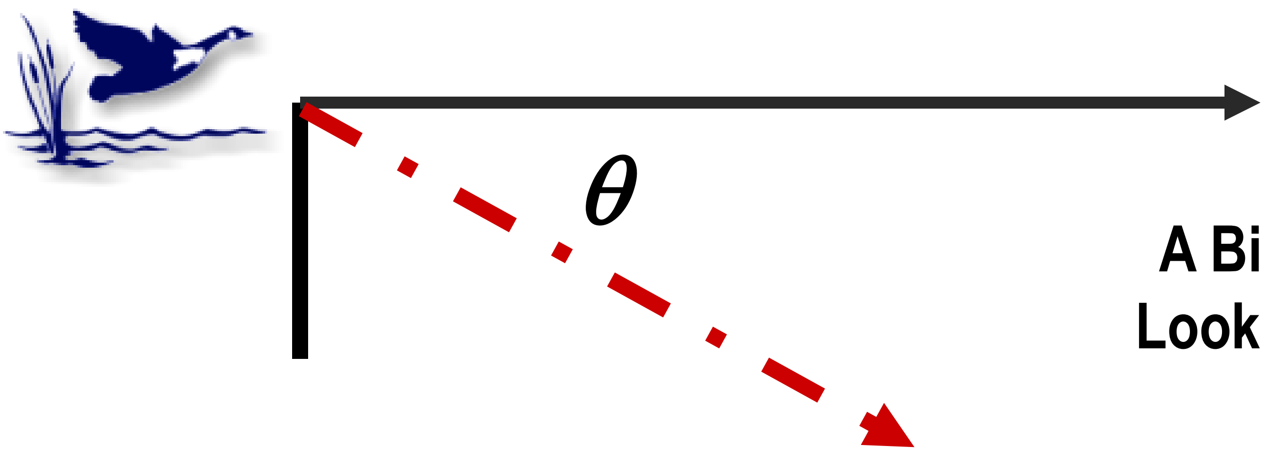
Angle of Elevation

- The angle between the line of sight (line joining the eye of an object) & the horizontal plane with the object **ABOVE** the horizontal plane.



Angle of Depression

- The angle between the line of sight (line joining the eye of an object) & the horizontal plane with the object **BELOW** the horizontal plane.



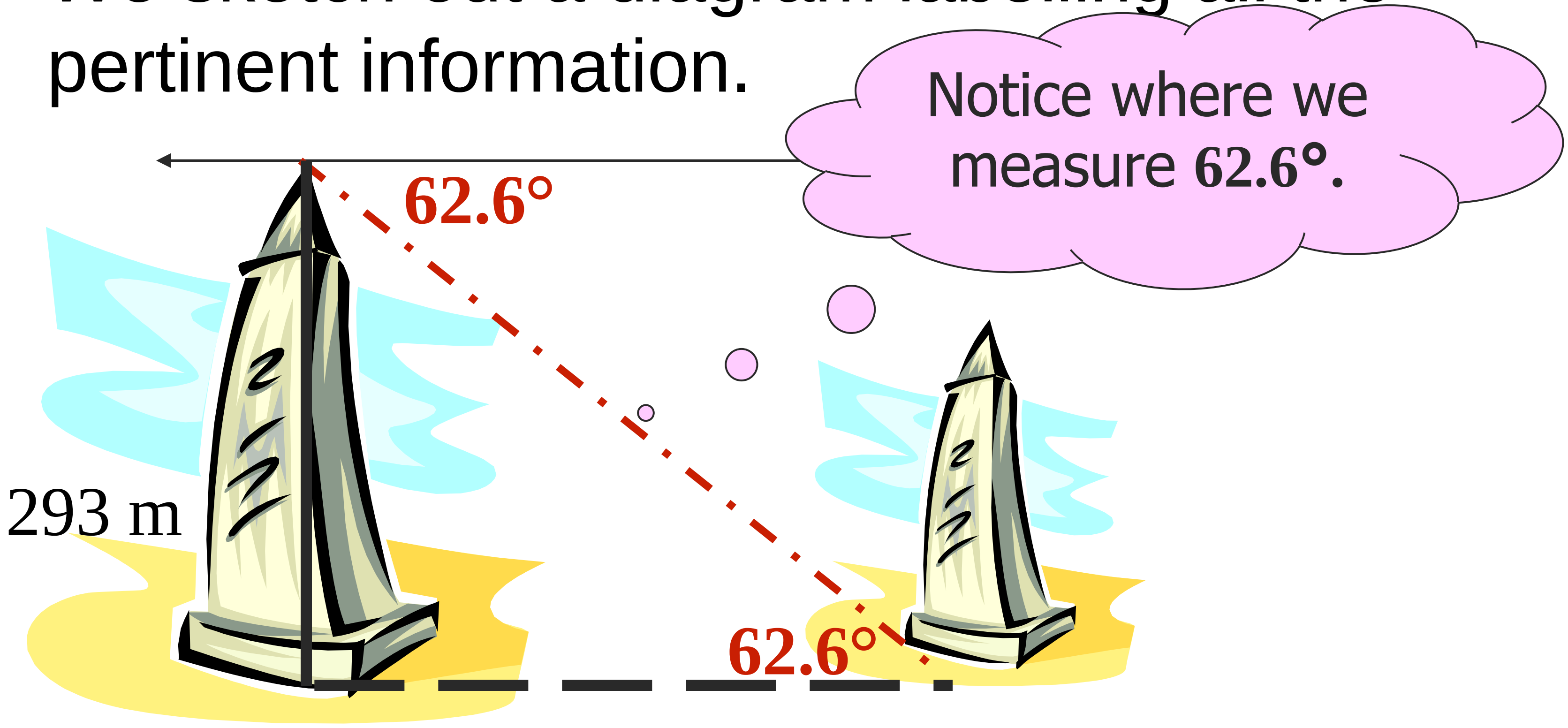
A Bird's View Looking Down!

Working with Angle of Depression

- In the diagram below, an observer on top of building A (293 m high) measures the angle of depression to the bottom of B as 62.6°. How far away is building B?

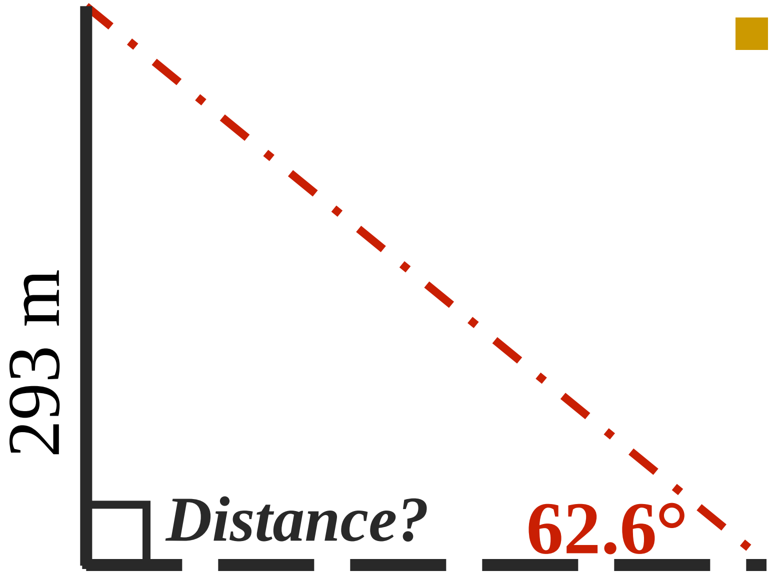
Working with Angle of Depression

- We sketch out a diagram labelling all the pertinent information.



Working with Angle of Depression

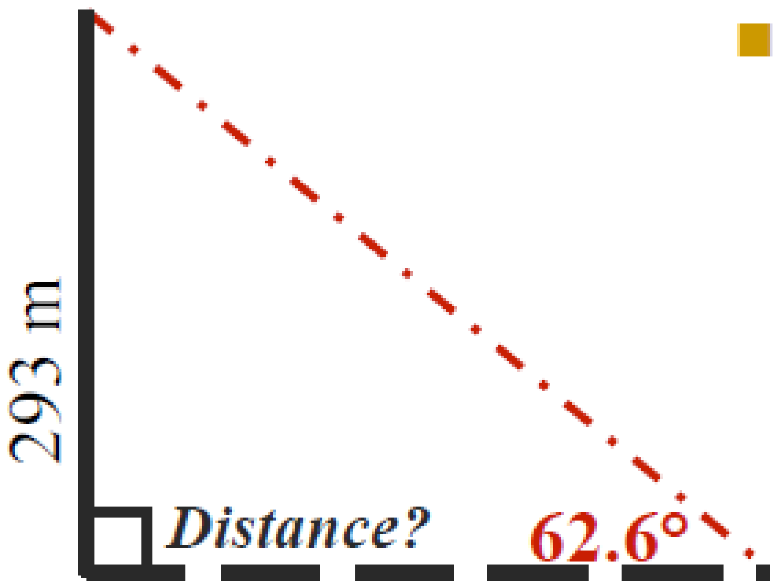
- If we remove the buildings, a right angle triangle has been created.



- We solve for ***distance*** having identified the appropriate trigonometric ratio.

Working with Angle of Depression

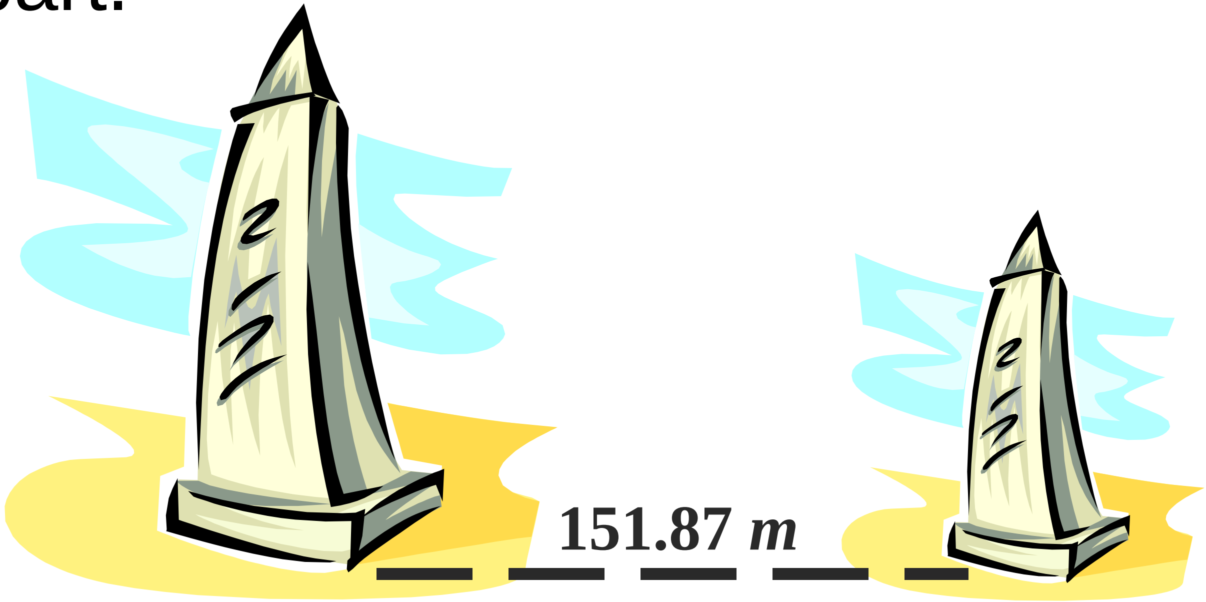
- If we remove the buildings, a right angle triangle has been created.



- We solve for ***distance*** having identified the appropriate trigonometric ratio.

Working with Angle of Depression

- Therefore, the buildings are 151.87 m apart.



?

Ex 4.5 q 6-9 & 17-20

6. $\sin 13.0^\circ = \dfrac{h}{1.25}$

$h = 1.25 \sin 13.0^\circ = 0.281 \text{ m}$

17. $\theta = \tan^{-1} \dfrac{6.0}{100} = 3.4^\circ$

A

6. $\sin 13.0^\circ = \dfrac{h}{1.25}$

$h = 1.25 \sin 13.0^\circ = 0.281 \text{ m}$

7. $25.0 \text{ ft} \cdot \dfrac{12 \text{ in.}}{1 \text{ ft}} = 300 \text{ in.}, \ \theta = \tan^{-1}\dfrac{2.00}{300} = 0.4^\circ$

8. $\tan A = \dfrac{4.63}{12.60}$

$A = \tan^{-1}\left(\dfrac{4.63}{12.60}\right) = 20.2^\circ$

9. $\cos 76.67^\circ = \dfrac{196.0}{h}$

$h = \dfrac{196.0}{\cos 76.67^\circ} = 850.1 \text{ cm}$

17. $\theta = \tan^{-1}\dfrac{6.0}{100} = 3.4^\circ$

18. $\dfrac{360^\circ}{8 \text{ sides}} = 45^\circ, \text{ so } \dfrac{45^\circ}{2} = 22.5^\circ$

$\sin 22.5 = \dfrac{0.375}{x}$

$x = \dfrac{0.375}{\sin 22.5^\circ} = 0.9799$

$2x = 1.96 \text{ m}$

19. $\tan \alpha = \dfrac{6.75}{15.5}; \ \alpha = \tan^{-1}\left(\dfrac{6.75}{15.5}\right) = 23.4^\circ$

20. $\sin 20.0^\circ = \dfrac{h}{12.5}$

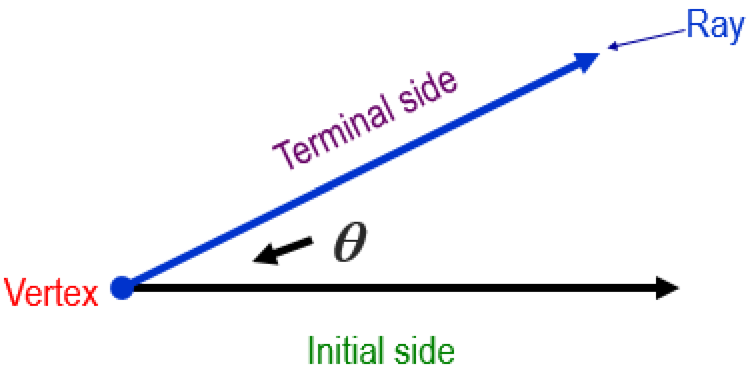
$h = 12.5 \sin 20.0^\circ = 4.3 \text{ ft}$

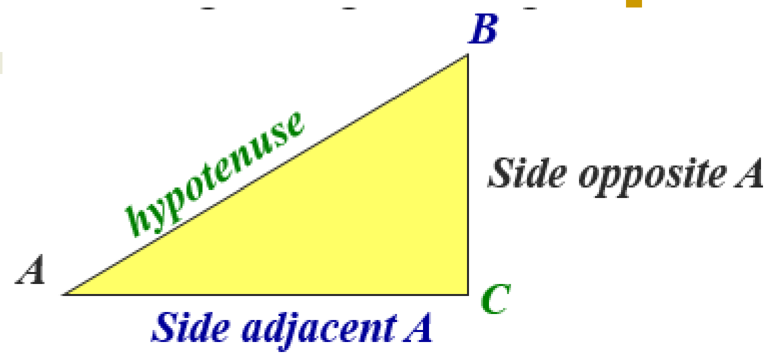
Height of the light above the street:

$4.3 + 28.0 = 32.3 \text{ ft}$

Summary below

M2 W1 Ch4

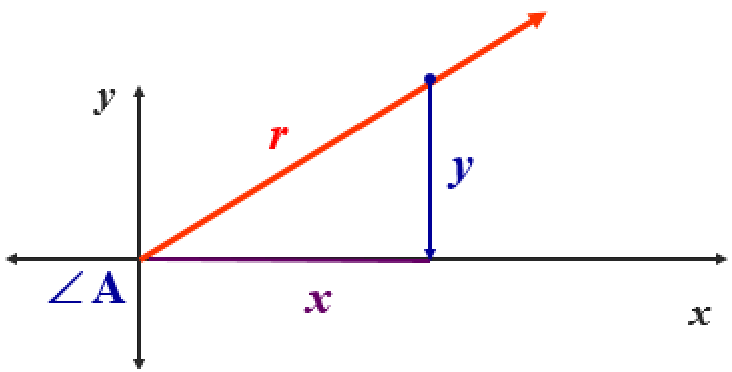




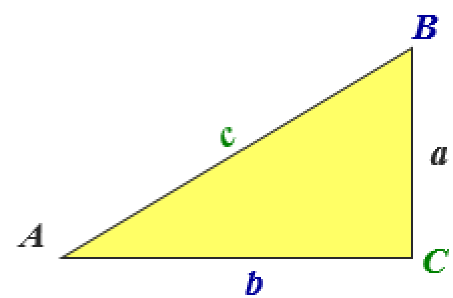
Coterminal
Positive = θ + 360
Negative = θ - 360

Pythagoras

SOH CAH TOA
S=O/H, C=A/H, T=O/A

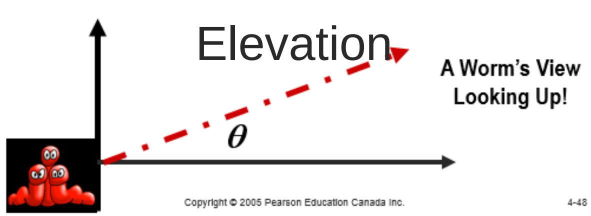


360° in 2π rads

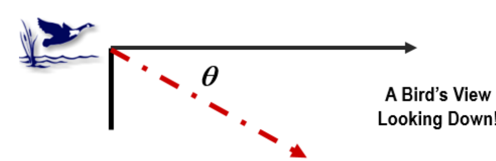


- Sine of θ: $\sin\theta = \frac{y}{r}$
- Cosecant of θ: $\csc\theta = \frac{r}{y}$

Elevation



csc A = 1/sin A
sec A = 1/cos A
cot A = 1/tan A



Depression

4-58

Ch. 5.4: Solving Systems of Linear Equations in Two Unknowns Algebraically

- Systems of linear equations can be solved algebraically to obtain exact solutions.
- These techniques include:
  - Solution by Substitution
  - Solution by Addition or Subtraction

Solution of Two Linear Equations by Substitution

1. Solve one equation for one of the unknowns.
2. Substitute this solution into the other equation to obtain a linear equation in one unknown.
3. Solve the resulting equation for the value of the unknown it contains.
4. Substitute this value into the equation of step 1 and solve for the other unknown.
5. Check the values in both original equations.

Example 1

- Solve the following system of linear equations in two unknowns by substitution.

1 $$4x + 3y = 18$$

2 $$x + 5y = 13$$

Solution 1

$$4x + 3y = 18 \quad (1)$$
$$x + 5y = 13 \quad (2)$$

1. Solve one equation for one of the unknowns.

$$x = 13 - 5y$$

2. Substitute this solution into the other equation to obtain a linear equation in one unknown.

$$4(13 - 5y) + 3y = 18$$

Solution 1 (*continued*)

1. $4x + 3y = 18$
2. $x + 5y = 13$

3. Solve the resulting equation for the value of the unknown it contains.

$$y = 2$$

4. Substitute this value into the equation of step 1 and solve for the other unknown.

$$x + 5y = 13$$

$$x + 5(2) = 13$$

$$x = 3$$

Answer: **(3, 2)**

Solution 1 *(continued)*

1. $4x + 3y = 18$
2. $x + 5y = 13$

5. Check the values in both original equations.

1.
$$4x + 3y \stackrel{?}{=} 18$$
$$4(3) + 3(2) \stackrel{?}{=} 18$$
$$12 + 6 \stackrel{?}{=} 18$$
$$18 \stackrel{\checkmark}{=} 18$$

2.
$$x + 5y \stackrel{?}{=} 13$$
$$3 + 5(2) \stackrel{?}{=} 13$$
$$13 \stackrel{\checkmark}{=} 13$$

- The solution checks in both equations.
- This is an independent system of linear equations.

ANOTHER METHOD
Solution of Two Linear Equations by Addition or Subtraction

1. If not already so, write the equations in the form $a_1x + b_1y = c_1$

   $$a_2x + b_2y = c_2$$

2. If necessary, multiply all terms of each equation by a constant chosen so that the coefficients of one unknown will be numerically the same in both equations. (They can have the same or different signs.)

Solution of Two Linear Equations by Addition or Subtraction (*continued*)

3. (a) If the numerically equal coefficients have ***different*** signs, ***add*** the terms on each side of the resulting equations.

   (b) If the numerically equal coefficients have the ***same*** sign, ***subtract*** the terms on each side of the resulting equations.
4. Solve the resulting linear equation in the other unknown.

Solution of Two Linear Equations by Addition or Subtraction (*continued*)

5. Substitute this value into one of the original equations to find the value of the other unknown.
6. Check by substituting both values into both original equations.

Example 2

- Solve the following system of linear equations in two unknowns by addition or subtraction.

1. $2x + 7y = 5$
2. $3x + 8y = 20$

$$
\begin{aligned}
\textbf{1}\quad & 2x + 7y = 5 \\
\textbf{2}\quad & 3x + 8y = 20
\end{aligned}
$$

N

Solution 2

1. We note the equations are in the correct form.
2. Multiply Equation 1 by **3** and Equation 2 by **2** to give the same first term.

$$2x + 7y = 5 \quad \times 3$$

$$3x + 8y = 20 \quad \times 2$$

$$6x + 21y = 15$$

$$6x + 16y = 40$$

Solution 2 *(continued)*

$$2x + 7y = 5 \quad (1)$$
$$3x + 8y = 20 \quad (2)$$

3. Since the signs of the 1st term are the same, we subtract each term.

$$6x + 21y = 15$$
$$-\ 6x - 16y = -40$$
$$5y = -25$$

4. Solve the resulting linear equation in the other unknown.

$$y = -5$$

Solution 2 *(continued)*

$$\text{①}\ 2x + 7y = 5$$
$$\text{②}\ 3x + 8y = 20$$

5. Substituting into Equation 1 to find ***x***.

$$2x + 7(-5) = 5$$
$$x = 20$$

6. Checking in both equations, the answer is: **(20, -5)**

Summary

- Be sure the systems of linear equations are in general form.
- If not, they must be rearranged into that form.
- Always check your work by substituting your answers into the original equations.

?

Ex 5.4 p151 5,9,13 15,19,23

In Exercises 5–14, solve the given systems of equations by the method of elimination by substitution.

5. $x = y + 3$
   $x - 2y = 5$
6. $x = 2y + 1$
   $2x - 3y = 4$
7. $p = V - 4$
   $V + p = 10$
8. $y = 2x + 10$
   $2x + y = -2$
9. $x + y = -5$
   $2x - y = 2$
10. $3x + y = 1$
    $3x - 2y = 16$
11. $2x + 3y = 7$
    $6x - y = 1$
12. $2s + 2t = 1$
    $4s - 2t = 17$
13. $33x + 2y = 34$
    $40y = 9x + 11$
14. $3A + 3B = -1$
    $5A = -6B - 1$

In Exercises 15–24, solve the given systems of equations by the method of elimination by addition or subtraction.

15. $x + 2y = 5$
    $x - 2y = 1$
16. $x + 3y = 7$
    $2x + 3y = 5$
17. $2x - 3y = 4$
    $2x + y = -4$
18. $R - 4r = 17$
    $3R + 4r = 3$
19. $12t + 9y = 14$
    $6t = 7y - 16$
20. $3x - y = 3$
    $4x = 3y + 14$
21. $v + 2t = 7$
    $2v + 4t = 9$
22. $3x - y = 5$
    $-9x + 3y = -15$
23. $2x - 3y - 4 = 0$
    $3x + 2 = 2y$
24. $3i_1 + 5 = -4i_2$
    $3i_2 = 5i_1 - 2$

A

5. (1) $x = y + 3$

(2) $x - 2y = 5$

$(y+3) - 2y = 5$ substitute x from (1) into (2)

$-y = 2$

$y = -2$ substitute -2 for y in (1)

$x = -2 + 3 = 1$

9. (1) $x + y = -5,\ y = -x - 5$

(2) $2x - y = 2$

$2x - (-x - 5) = 2$ substitute y from (1) into (2)

$3x = -3$

$x = -1$

$-1 + y = -5$ substitute -1 for x in (1)

$y = -4$

13. (1) $33x + 2y = 34 \Rightarrow y = -\frac{33}{2}x + 17$

(2) $40y = 9x + 11$

$40\left(-\frac{33}{2}x + 17\right) = 9x + 11$ substitute y from (1) in (2)

$-660x + 680 = 9x + 11$

$-669x = -669$

$x = 1$

15.

$$\begin{aligned} x + 2y &= 5 \\ \underline{x - 2y} &\underline{= 1} \\ 2x &= 6 \\ x &= 3 \\ 3 + 2y &= 5 \\ 2y &= 2 \\ y &= 1;\ (3, 1) \end{aligned}$$

19. (1) $12t + 9y = 14 \Rightarrow \quad 12t + 9y = 14$

(2) $6t = 7y - 16 \Rightarrow \underline{-12t + 14y = 32}$ add

$23y = 46$

$y = 2$

$12t + 9(2) = 14$ substitute 2 for y in (1)

$12t = -4$

$t = -\frac{1}{3}$

23.

(1)	$2x - 3y - 4 = 0$	
(2)	$3x + 2 = 2y$	put (1) in standard form
(3)	$2x - 3y = 4$	recopy (2)
(4)	$3x - 2y = -2$	put (2) in standard form
(5)	$6x - 9y = 12$	(3) multiplied by 3
(6)	$\underline{-6x + 4y = 4}$	(4) multiplied by -2
	$-5y = 16$	
	$y = -\frac{16}{5}$	

from (1) $x = \frac{3y+4}{2} = \frac{3\left(-\frac{16}{5}\right)+4}{2} = -\frac{14}{5}$,

$\left(-\frac{14}{5}, -\frac{16}{5}\right)$ is the solution.

Ch. 5.5: Solving Systems of Linear Equations in Two Unknowns by Determinants

- To solve a system of linear equations algebraically has involved working with the variables, x & y.
- When working with determinants, we eliminate the use of x & y and work simply with the coefficients and constant, c.

Solving Systems of Linear Equations in Two Unknowns by Determinants

- Given:
$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

MUST set up equations like this

- The solution for ***x*** & ***y*** can be shown to be:

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \qquad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

Determinant of the 2nd order

P153

Definition of the Determinant of the Second Order

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

rows

columns

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Principal diagonal - ***Secondary diagonal***

Cramer's Rule

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

- According to Cramer's Rule, we can solve for ***x*** in the original equation by setting up the following equation:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

- *Notice, that the denominator is the determinant.*

Cramer's Rule

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

- To calculate the value of ***y***, we can use:

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

- *The value of* ***x*** *can also be simply substituted into an equation.*

- *The denominator remains the same.*

Cramer's Rule

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

EXAMPLE 4 Solving a system using Cramer's rule

Solve the following system of equations by determinants. Numbers are approximate.

$$5.3x + 7.2y = 4.5$$
$$3.2x - 6.9y = 5.7$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

constants (numerator, first column); coefficients (denominator)

$$x = \frac{\begin{vmatrix} 4.5 & 7.2 \\ 5.7 & -6.9 \end{vmatrix}}{\begin{vmatrix} 5.3 & 7.2 \\ 3.2 & -6.9 \end{vmatrix}} = \frac{4.5(-6.9) - 5.7(7.2)}{5.3(-6.9) - 3.2(7.2)} = \frac{-72.09}{-59.61} = 1.2$$

constants (numerator, second column); coefficients (denominator)

$$y = \frac{\begin{vmatrix} 5.3 & 4.5 \\ 3.2 & 5.7 \end{vmatrix}}{\begin{vmatrix} 5.3 & 7.2 \\ 3.2 & -6.9 \end{vmatrix}} = \frac{5.3(5.7) - 3.2(4.5)}{-59.61} = \frac{15.81}{-59.61} = -0.27$$

The calculations can all be done on a calculator, using the following procedure: (1) Evaluate the denominator and store this value; (2) divide the value of each numerator by the value of the denominator, storing the values of x and y; (3) use the stored values of x and y to check the solution; (4) round off results (if the numbers are approximate).

For this system, we store the value of the denominator, $D = -59.61$, in order to calculate x and y, stored as X and Y. The check is shown in the calculator display in Fig. 5.33. ■

? Ex 5.5 p157 17, 21, 25

In Exercises 17–26, solve the given systems of equations by determinants. (These are the same as those for Exercises 15–24 of Section 5.4.)

17. $x + 2y = 5$
$x - 2y = 1$

18. $x + 3y = 7$
$2x + 3y = 5$

19. $2x - 3y = 4$
$2x + y = -4$

20. $R - 4r = 17$
$3R + 4r = 3$

21. $12t + 9y = 14$
$6t = 7y - 16$

22. $2x + y = 4$
$19x = 80 - 10y$

23. $v + 2t = 7$
$2v + 4t = 9$

24. $3x - y = 5$
$-9x + 3y = -15$

25. $2x - 3y - 4 = 0$
$3x + 2 = 2y$

26. $3i_1 + 5 = -4i_2$
$3i_2 = 5i_1 - 2$

Г A

17. $x + 2y = 5$
$x - 2y = 1$

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}} = \frac{(5)(-2) - (1)(2)}{(1)(-2) - (1)(2)} = \frac{-10 - 2}{-2 - 2} = 3$$

$$y = \frac{\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}}{-4} = \frac{(1)(1) - (1)(5)}{-4} = \frac{1 - 5}{-4} = 1$$

21. Rewrite the system with both equations in standard form.

$12t + 9y = 14$
$6t - 7y = -16$

$$t = \frac{\begin{vmatrix} 14 & 9 \\ -16 & -7 \end{vmatrix}}{\begin{vmatrix} 12 & 9 \\ 6 & -7 \end{vmatrix}} = \frac{46}{-138} = -\frac{1}{3},$$

$$y = \frac{\begin{vmatrix} 12 & 14 \\ 6 & -16 \end{vmatrix}}{-138} = \frac{-276}{-138} = 2$$

25. -3 Rewrite the system with both equations in standard form.

$2x - 3y = 4$
$3x - 2y = -2$

$$x = \frac{\begin{vmatrix} 4 & -3 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}} = -2.8 = -\frac{14}{5}$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}} = -3.2 = \frac{-16}{5}$$

M2 Ch5.4& 5.5

- Be sure the systems of linear equations are in general form.
- If not, they must be rearranged into that form.
- Always check your work by substituting your answers into the original equations.

3 Methods

- Solution by Substitution
- Solution by Addition or Subtraction
- Cramer's Rule

Cramer's Rule

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Chapter 8

Trigonometric Functions of Any Angle

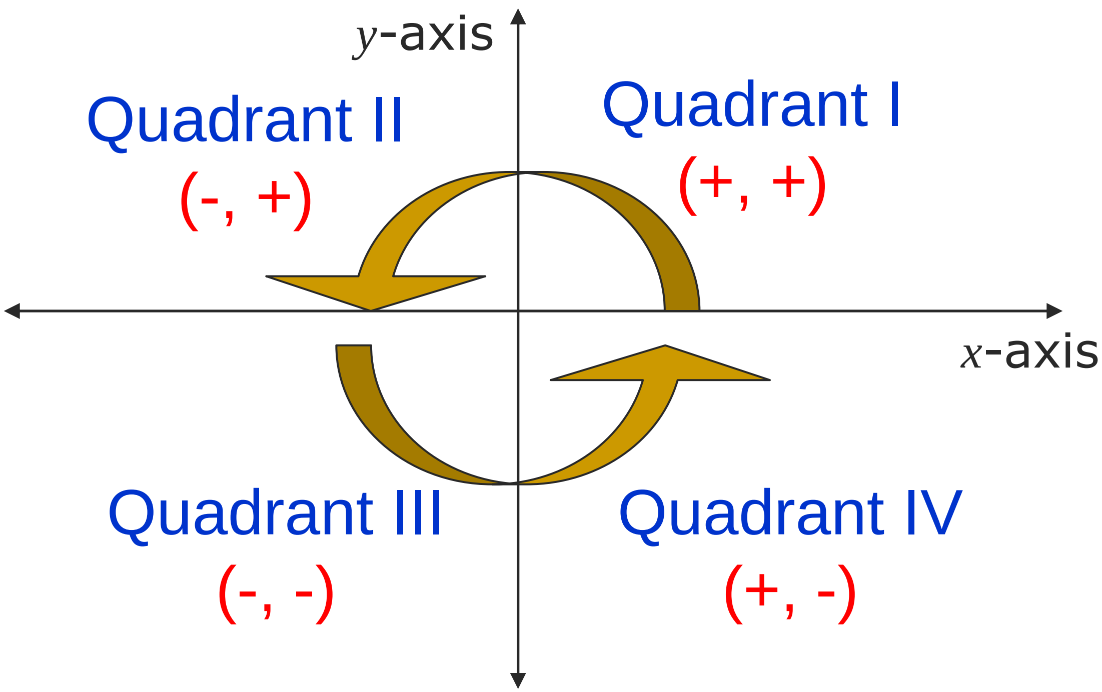
Learning Outcomes

- At the end of this chapter the student will:
  - Determine the values of the trigonometric functions for angles in any quadrant of the Cartesian plane.
  - Calculate angles in radians
  - Apply the concept of radian measure when calculating arc length, area of a sector of a circle and angular velocity.

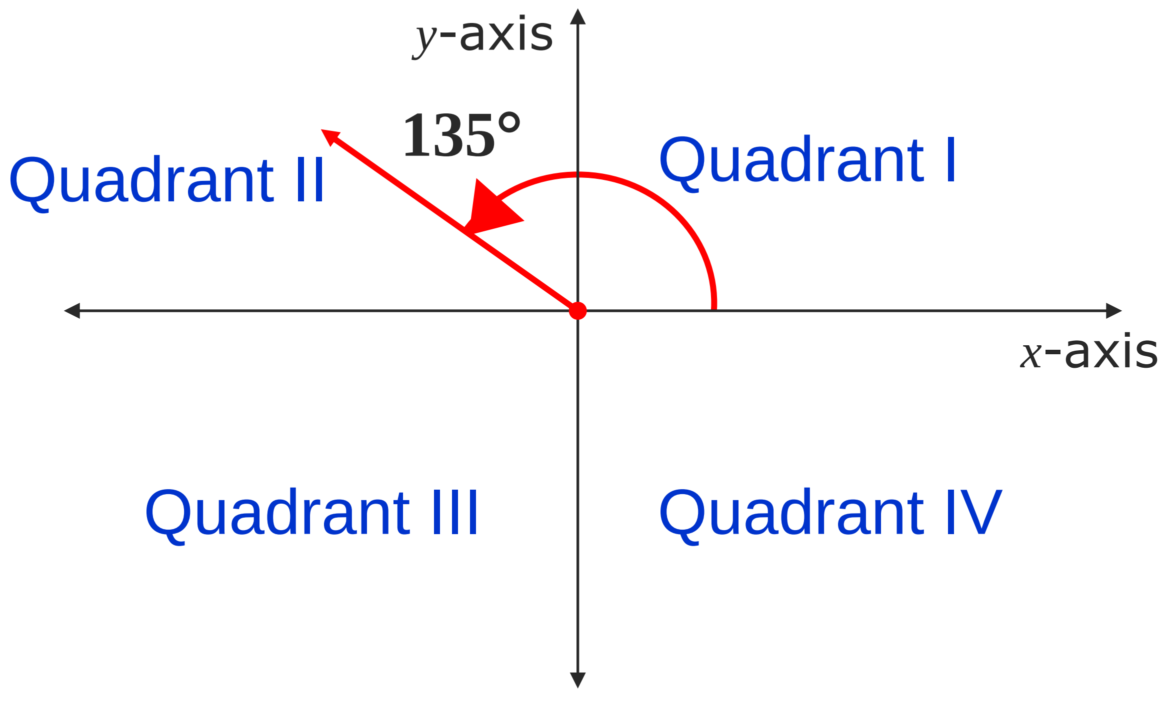
Ch. 8.1: Signs of the Trigonometric Functions

- Previously calculated for acute angles.
- Now we need methods to solve:
  - obtuse angles,
  - negative angles,
  - angles greater than one revolution, and,
  - angles with terminal sides on the coordinate axis.

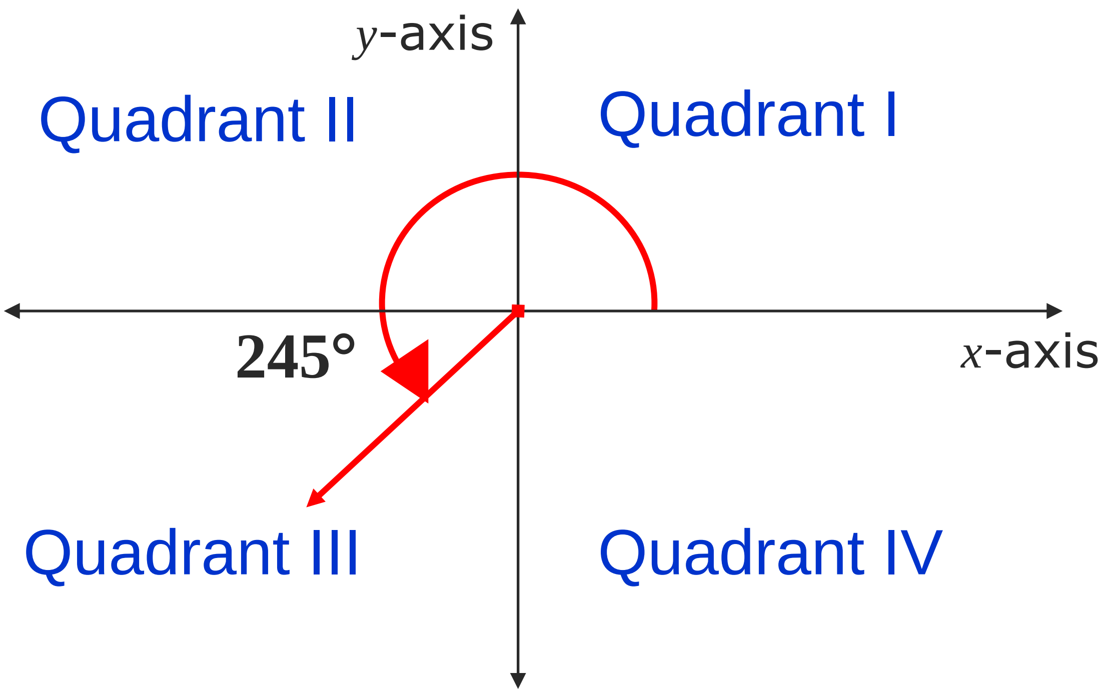
Mapping the Cartesian (*xy*) plane.



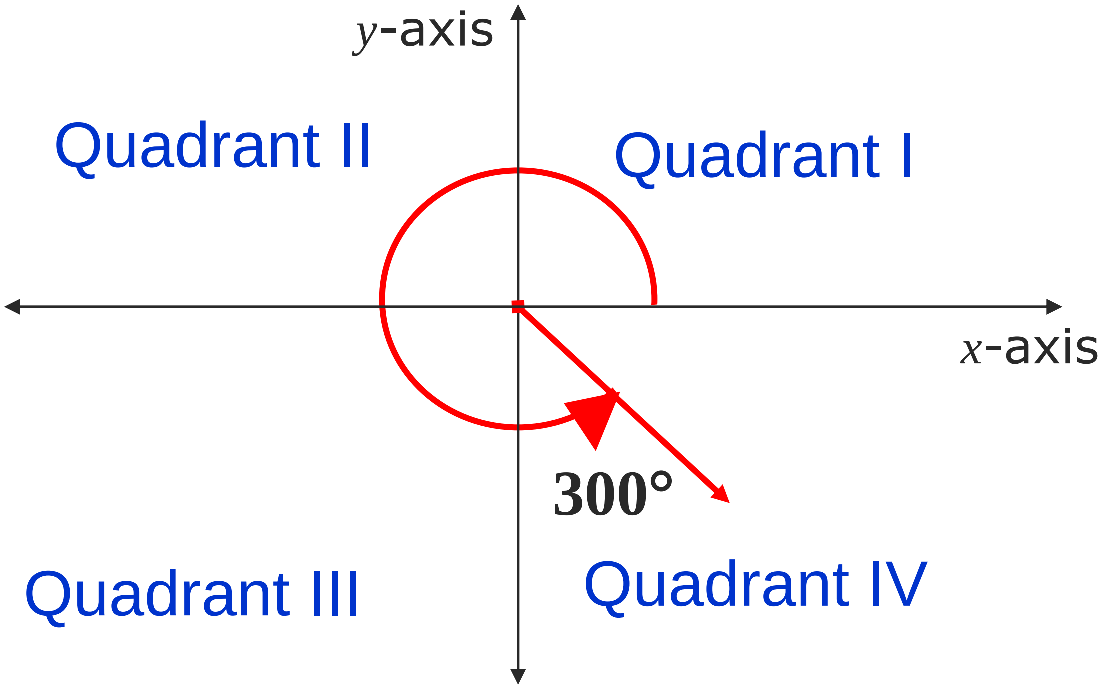
An Angle in QII



An Angle in QIII



An Angle in QIV



The CAST Rule

y-axis

x-axis

S

QII

Only the **Sine** function is positive.

A

QI

All trig functions are positive.

T

QIII

Only the **Tangent** function is positive.

C

QIV

Only the **Cosine** function is positive.

Useful relationships

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Summary

- We can use our calculator to find the trigonometric functions of an angle in any quadrant.
- Angles found in QII, QIII and QIV have corresponding reference angles.
- The CAST Rule will identify the sign of any trigonometric function.

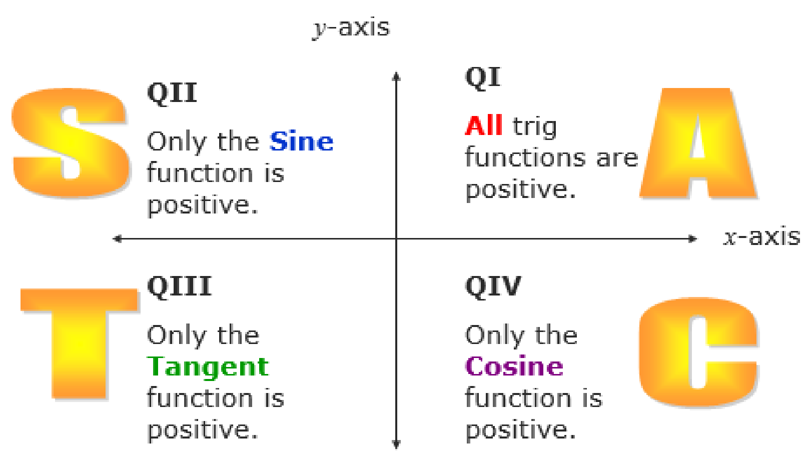
$\csc(\theta) = 1/\sin(\theta)$	$\sec(\theta) = 1/\cos(\theta)$	$\cot(\theta) = 1/\tan(\theta)$

?

Ex 8.1 3-6 & 29-32

3. sin 36° is positive since 36° is in QI where sine is positive.
cos 120° is negative since 120° is in QII where cosine is negative.

30. tan θ is positive and cos θ is negative
tan θ is positive in QI and QIII
cos θ is negative in QII and QIII. The terminal side of θ must lie in QIII to meet both conditions.



A

3. sin 36° is positive since 36° is in QI where sine is positive.

cos 120° is negative since 120° is in QII where cosine is negative.

4. tan 320° is negative since 320° is in QIV, where tan θ is negative.

sec 185° is negative since 185° is in QIII, where sec θ is negative.

5. csc 98° is positive since 98° is in QII, where csc θ is positive.

cot 82° is positive since 82° is in QI, where cot θ is positive.

6. cos 260° is negative since 260° is in QIII, where cos θ is negative.

csc 290° is negative since 290° is in QIV, where csc θ is negative.

30. tan θ is positive and cos θ is negative

tan θ is positive in QI and QIII

cos θ is negative in QII and QIII. The terminal side of θ must lie in QIII to meet both conditions.

31. sec θ is negative and cot θ is negative

sec θ is negative in QII and QIII.

cot θ is negative in QII and QIV.

The terminal side of θ must lie in QII to meet both conditions.

32. cos θ is positive and csc θ is negative

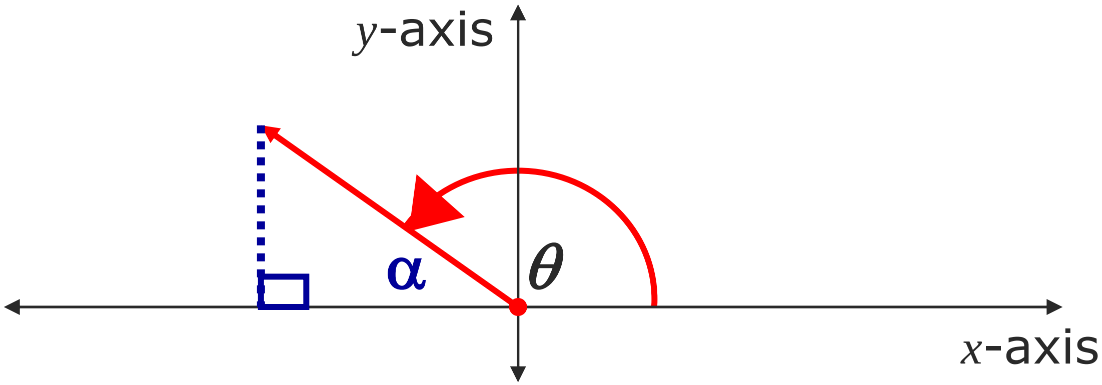
cos θ is positive in QI and QIV.

csc θ is negative in QIII and QIV. The terminal side of θ must lie in QIV to meet both conditions.

Ch. 8.2: Trigonometric Functions of Any Angle

- Given an angle in any quadrant, we can determine the trigonometric functions of that angle.
- Given the trigonometric function(s), we can determine the angle in any quadrant.
- When determining the angle in any quadrant, we need to make use of the reference angle.

Reference Angles



- The ***reference angle, α***, of a given angle is the acute angle formed by the terminal side of the angle and the ***x***-axis.

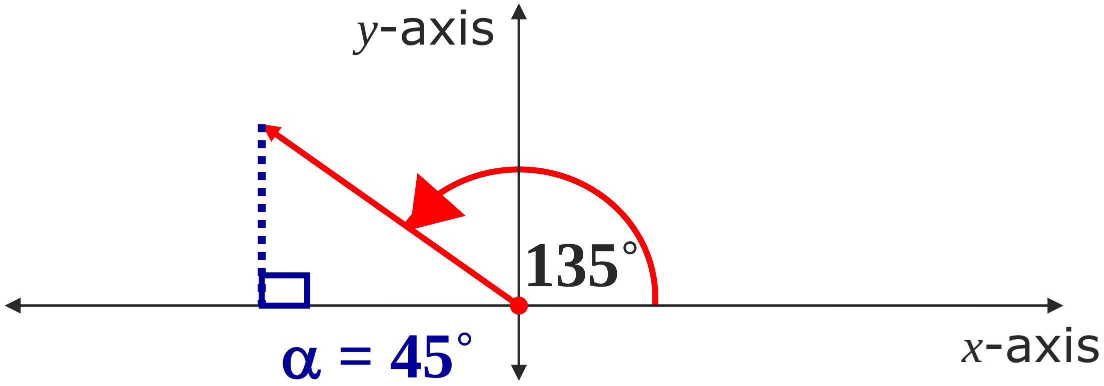
Reference Angles

- The general formula for finding the reference angle is:

$$F(\theta_2) = \pm F\left(180° - \theta_2\right) = \pm F\left(\alpha\right)$$

- The sign used depends on the sign of the function in the second quadrant.
- F represents any trigonometric function

Example



- Angle **45°** is the reference angle to **135°**.

$$F(\theta_2) = \pm F\left(180° - \theta_2\right) = \pm F\left(\alpha\right)$$

$$F(45) = \pm F\left(180° - 135°\right) = \pm F\left(\alpha\right)$$

Steps to Finding Trig Values in any Quadrant

1. Plot the given point.
2. Identify the reference angle & label the sides accordingly.
3. Find the reference angle using $\tan^{-1}$.
4. Add the appropriate degrees to obtain the angle looking for.
5. Either use the calculator to find the trig functions or use the given information. You may have to find the hypotenuse using

$$r = \sqrt{x^2 + y^2}$$

Finding the Reference Angle

- The required angle θ is found by using the reference angle as shown:

$$\theta = \alpha \text{ (Q I)}$$

$$\theta = 180^\circ - \alpha \text{ (Q II)}$$

$$\theta = 180^\circ + \alpha \text{ (Q III)}$$

$$\theta = 360^\circ - \alpha \text{ (Q IV)}$$

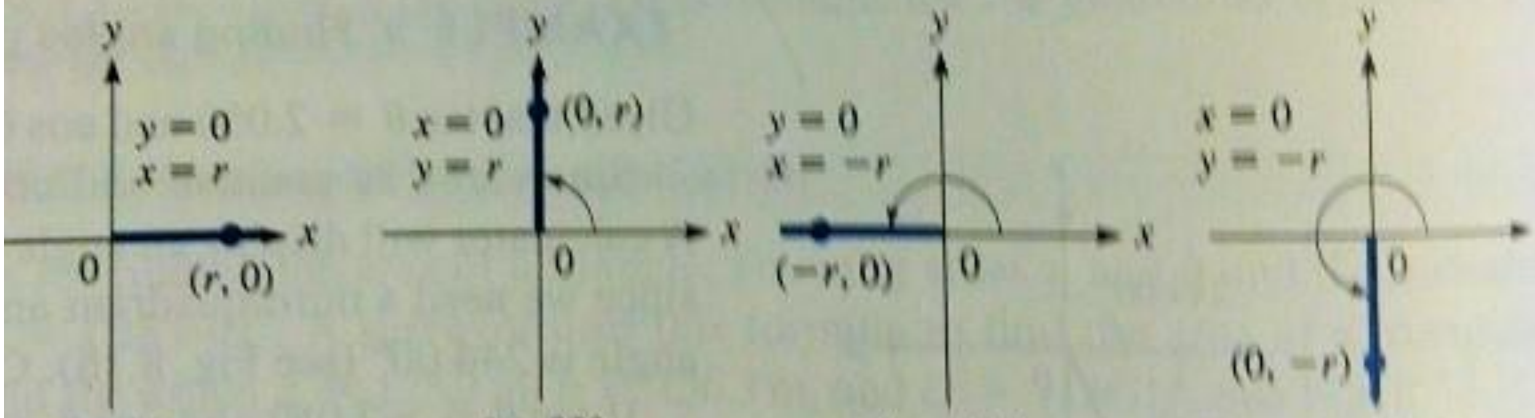
The Quadrantal Angles

- These angles are represented by the x and y axes.
- They are 0°, 90°, 180°, 270° and 360°.
- Their terminal side lies on a coordinate axis.

What are the trig values for these angles?

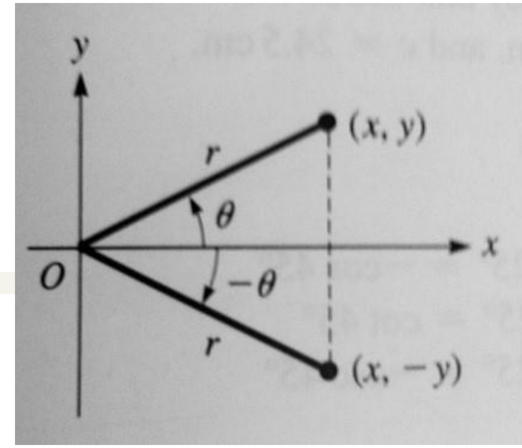
Fig 8.18

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0.000	1.000	0.000	undef.	1.000	undef.
90°	1.000	0.000	undef.	0.000	undef.	1.000
180°	0.000	−1.000	0.000	undef.	−1.000	undef.
270°	−1.000	0.000	undef.	0.000	undef.	−1.000
360°	Same as the functions of 0° (same terminal side)					

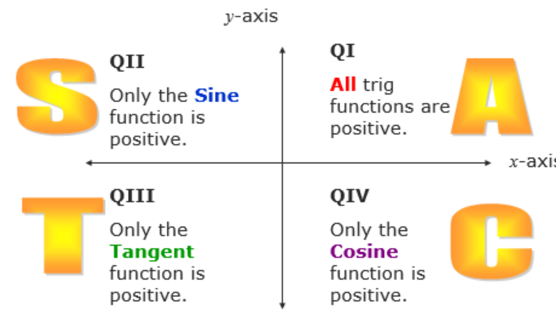


$\theta = 0°$ (a) $\theta = 90°$ (b) $\theta = 180°$ (c) $\theta = 270°$ (d)

Negative Angles



- To find the values of the functions of negative angles, we can use functions of corresponding positive angles, ***if we use the correct sign***.



$$\sin(-\theta) = -\sin\theta$$

$$\csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

csc(θ) = 1/sin(θ)	sec(θ) = 1/cos(θ)	cot(θ) = 1/tan(θ)

Ex 8.2 questions 5-8, 41-44

5. $\sin 160^\circ = \sin\left(180^\circ - 160^\circ\right) = \sin 20^\circ$

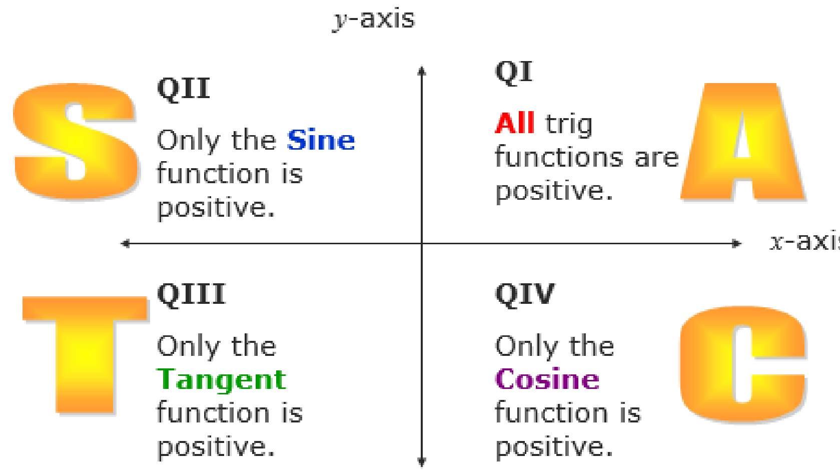
$\cos 220^\circ = \cos\left(180^\circ + 40^\circ\right) = -\cos 40^\circ$

41. $\sin\theta = -0.5736;\ \theta_{ref} = \sin^{-1}\left(-0.5736\right) = -35^\circ$

Since $\cos\theta > 0$, θ is in QIV.

$\theta = 325^\circ + k \times 360^\circ$ where $k = 0, \pm 1, \pm 2, \cdots$

$\tan\theta = -0.7003$



$\theta = \alpha$ (Q I)

$\theta = 180^\circ - \alpha$ (Q II)

$\theta = 180^\circ + \alpha$ (Q III)

$\theta = 360^\circ - \alpha$ (Q IV)

csc(θ) = 1/sin(θ)	**sec(θ) = 1/cos(θ)**	**cot(θ) = 1/tan(θ)**

A

6. $\tan 91^\circ = -\tan\left(180^\circ - 91^\circ\right) = -\tan 89^\circ$

$\sec 345^\circ = \sec\left(360^\circ - 345^\circ\right) = \sec 15^\circ$

7. $\tan 105^\circ = -\tan\left(180^\circ - 105^\circ\right) = -\tan 75^\circ$

$\csc 302^\circ = -\csc\left(360^\circ - 302^\circ\right) = -\csc 58^\circ$

8. $\cos 190^\circ = -\cos\left(190^\circ - 180^\circ\right) = -\cos 10^\circ$

$\cot 290^\circ = -\cot\left(360^\circ - 290^\circ\right) = -\cot 70^\circ$

42. $\cos\theta$ is positive, so θ is in QI or QIV. $\tan\theta$ is negative, so θ is in QII or QIV. θ is in QIV.

$\cos\theta = 0.422$; $\theta_{ref} = 65.0^\circ$; $\theta = 360^\circ - 65.0^\circ$

$= 295.0^\circ$

$\sin 295.0^\circ = -0.907$

43. $\tan\theta$ is negative, so θ is in QII or QIV. $\csc\theta$ is positive, so θ is in QI or QII. θ is in QII.

$\tan\theta = -0.809$; $\theta_{ref} = 39.0^\circ$; $\theta = 180^\circ - 39.0^\circ$

$= 141.0^\circ$

$\cos 141.0^\circ = -0.777$

44. $\sec\theta$ is positive, so θ is in QI or QIV. $\sin\theta$ is negative, so θ is in QIII or QIV. θ is in QIV.

$\sec\theta = 6.122$; $\theta_{ref} = 80.6^\circ$; $\theta = 360^\circ - 80.6^\circ$

$= 279.4^\circ$

$\cot 279.4^\circ = -0.1655$

Ch. 8.3: Radians

- Procedure for Converting Angle Measurements
  - To convert an angle measured in degrees to the same angle measured in radians, ***multiply the number of degrees by*** π rads/180°.
  - To convert an angle measured in radians to the same angle measured in degrees, ***multiply the number of radians by*** 180°/π rads.

?

Ex 8.3 q 5-8, 37-39

5. $15^\circ = \dfrac{\pi}{180}(15) = \dfrac{\pi}{12}$

$150^\circ = \dfrac{\pi}{180}(150) = \dfrac{5\pi}{6}$

37. $\sin\dfrac{\pi}{4} = \sin\left[\left(\dfrac{\pi}{4}\right)\left(\dfrac{180}{\pi}\right)\right] = \sin 45^\circ = 0.7071$

A

5. $15^\circ = \frac{\pi}{180}(15) = \frac{\pi}{12}$

$150^\circ = \frac{\pi}{180}(150) = \frac{5\pi}{6}$

6. $12^\circ = \frac{\pi}{180}(12) = \frac{12\pi}{180} = \frac{\pi}{15}$

$225^\circ = \frac{\pi}{180}(225) = \frac{225\pi}{180} = \frac{5\pi}{4}$

7. $75^\circ = \frac{\pi}{180}(75) = \frac{75\pi}{180} = \frac{5\pi}{12}$

$330^\circ = \frac{\pi}{180}(330) = \frac{330\pi}{180} = \frac{11\pi}{6}$

8. $36^\circ = \frac{\pi}{180}(36) = \frac{36\pi}{180} = \frac{\pi}{5}$

$315^\circ = \frac{\pi}{180}(315) = \frac{315\pi}{180} = \frac{7\pi}{4}$

37. $\sin\frac{\pi}{4} = \sin\left[\left(\frac{\pi}{4}\right)\left(\frac{180}{\pi}\right)\right] = \sin 45^\circ = 0.7071$

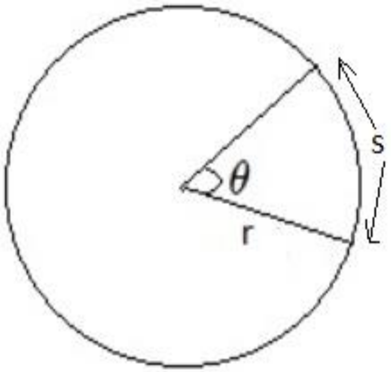
38. $\cos\frac{\pi}{6} = \cos\left[\left(\frac{\pi}{6}\right)\left(\frac{180}{\pi}\right)\right] = \cos 30^\circ = 0.8660$

39. $\tan\frac{5\pi}{12} = \tan\left[\left(\frac{5\pi}{12}\right)\left(\frac{180}{\pi}\right)\right] = \tan 75^\circ = 3.732$

Ch. 8.4: Applications of Radian Measure

- **In these applications, the angle that is used is in radians.**
- Be sure to convert any angle in degrees into radians before using these formulas.
  - Arc Length
  - Area of a Sector of a Circle
  - Angular Velocity

Arc Length



- The length of an arc on a circle is proportional to the central angle.

$$s = \theta r$$

- The length of arc of a complete circle is the circumference.

$$s = 2\pi r$$

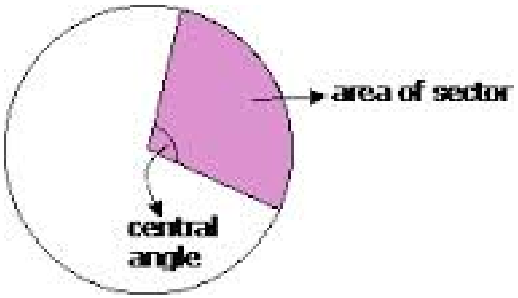
θ must be in radians.

Area of a Sector of a Circle

- The area of a sector of a circle is proportional to its central angle.

θ must be in radians.

$$A_{sector} = r^2\left(\frac{\theta}{2}\right)$$



Angular Velocity

- The relationship between the linear velocity ***v*** and the angular velocity **ω** of an object moving around a circle of radius ***r*** is represented as:

$$v = \omega r$$

ω is measured in radians per unit of time.

?

Ex 8.4 q5-8, q 30-31

5. $s = r\theta = (3.30)\left(\frac{\pi}{3}\right) = 3.46 \text{ in.}$

30. $v = \sqrt{2gh} = \sqrt{2(9.80)(4.80)} = \omega r$

$$\omega = \frac{v}{r} = \frac{\sqrt{2(9.80)(4.80)}}{13.8} = 0.703 \text{ rad/s}$$

Г A

5. $s = r\theta = (3.30)\left(\frac{\pi}{3}\right) = 3.46 \text{ in.}$

6. $s = \theta r = 2.65(21.2) = 56.2 \text{ cm}$

7. $\theta = 136\left(\frac{\pi}{180}\right) = 2.37 \text{ rad}$

$$r = \frac{s}{\theta} = \frac{1010}{2.37} = 426 \text{ mm}$$

8. $\theta = 73.61\left(\frac{\pi}{180}\right) = 1.285 \text{ rad}$

$$r = \frac{s}{\theta} = \frac{0.3456}{1.285} = 0.2689 \text{ ft}$$

$$A = \frac{1}{2}\theta r^2 = \frac{1}{2}(1.285)(0.2689)^2 = 0.0465 \text{ ft}^2$$

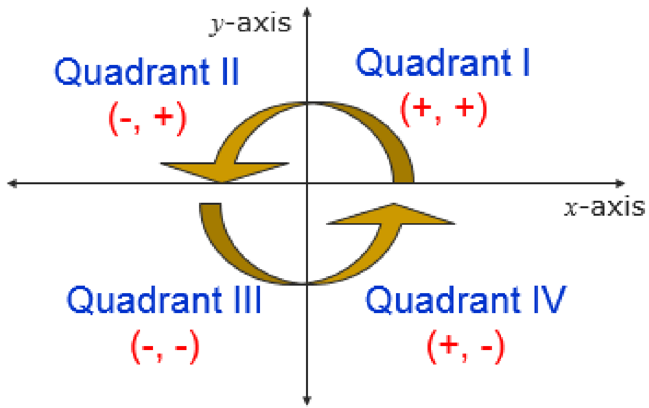
30. $v = \sqrt{2gh} = \sqrt{2(9.80)(4.80)} = \omega r$

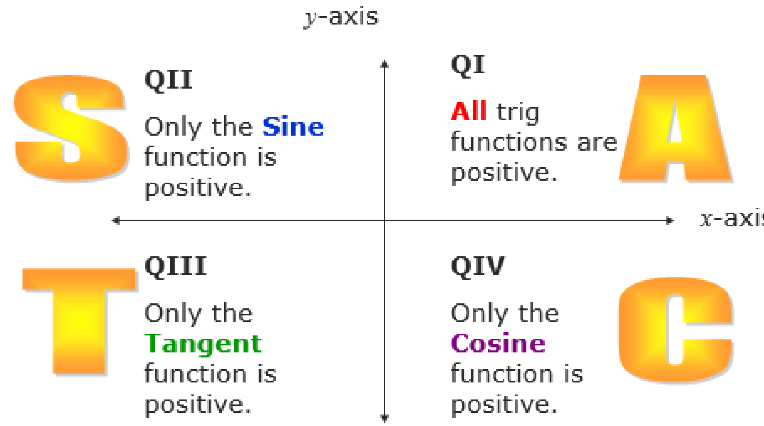
$$\omega = \frac{v}{r} = \frac{\sqrt{2(9.80)(4.80)}}{13.8} = 0.703 \text{ rad/s}$$

31. $\theta = 75.5^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{75.5\pi}{180};\ A = \frac{1}{2}\theta r^2$

$$r_1 = \frac{11.2}{2} + 2.50 = 8.10 \text{ m};\ r_2 = 5.60 \text{ m}$$

$$A_1 - A_2 = \frac{1}{2}\left(\frac{75.5\pi}{180}\right)\left(8.10^2 - 5.60^2\right) = 22.6 \text{ m}^2$$





$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

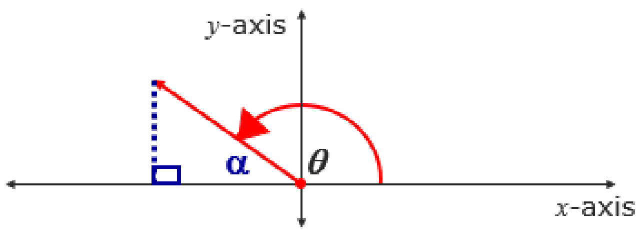
$$\cot\theta = \frac{1}{\tan\theta}$$

$\theta = \alpha$ (Q I)

$\theta = 180^\circ - \alpha$ (Q II)

$\theta = 180^\circ + \alpha$ (Q III)

$\theta = 360^\circ - \alpha$ (Q IV)



- The ***reference angle, α***, of a given angle is the acute angle formed by the terminal side of the angle and the *x*-axis.

$$s = \theta r \qquad A_{sector} = r^2\left(\frac{\theta}{2}\right)$$

$$v = \omega r$$

$$2\pi \ radians = 360 \ degrees$$

$$1 \, radian = \frac{180}{\pi} \ degrees$$

$$1 \, degree = \frac{\pi}{180} \ radians$$

M2 W2 Ch8

Chapter 9

Vectors and Oblique Triangles

Learning Outcomes Chapter 9

- I can add vectors by the tip to toe method
- I can add vectors by the parallelogram method
- I know how to multiply a vector by a scalar
- I can also take away vectors as A - B = A + (-B)

- I can split vectors up into x and y components
- I can add vectors by resolving the components.

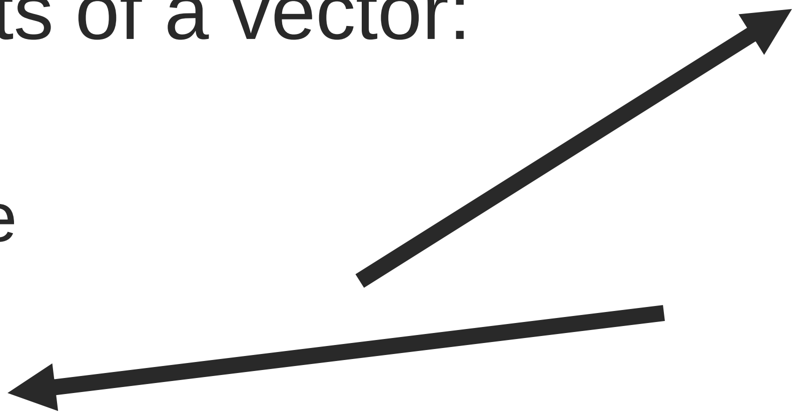
- I can use the cosine rule of triangles
- I can use the sine rule of triangles

- I can solve questions using the application of vectors.

Ch. 9.1: Introduction to Vectors

- Definition of a vector:
  -a directed line segment

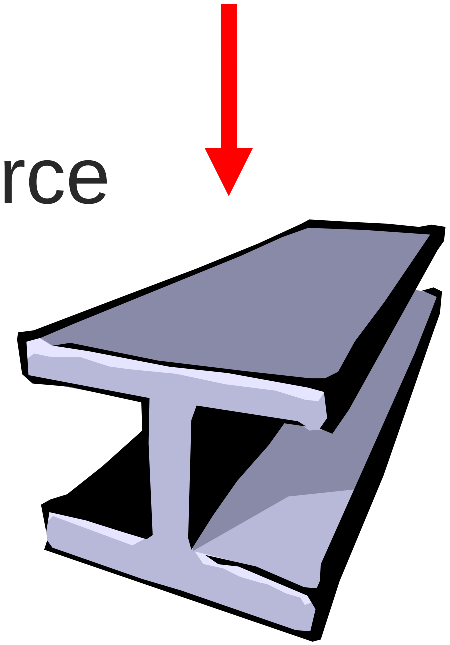
- Components of a vector:
  - Direction
  - Magnitude

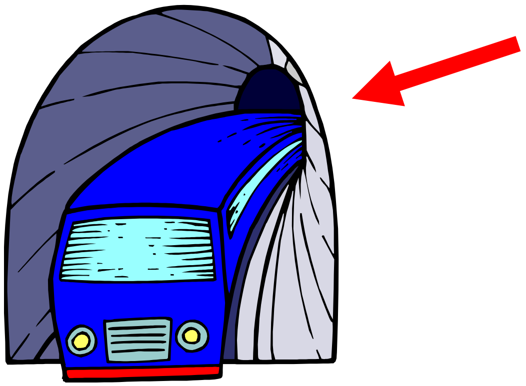


Vectors

- Applications of Vectors:

Force

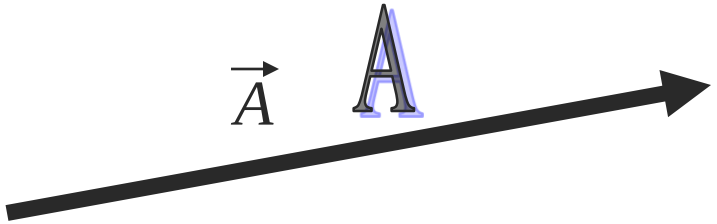




Velocity & acceleration

Describing Vectors

- How are vectors described?
  1. $\vec{A}$, or,
  2. in boldface: $\mathbf{A}$



Describing Vectors

- How is the magnitude or the size of a vector described?

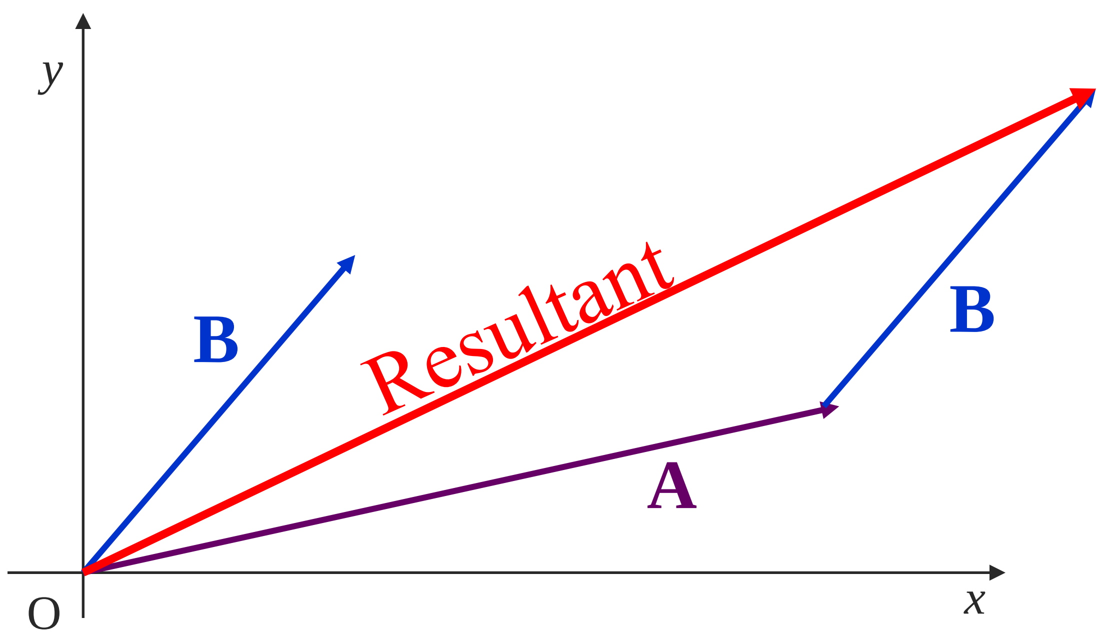
|AB| or ***AB***

Note the italicized lettering.

Note the boldface lettering within the absolute value bars.

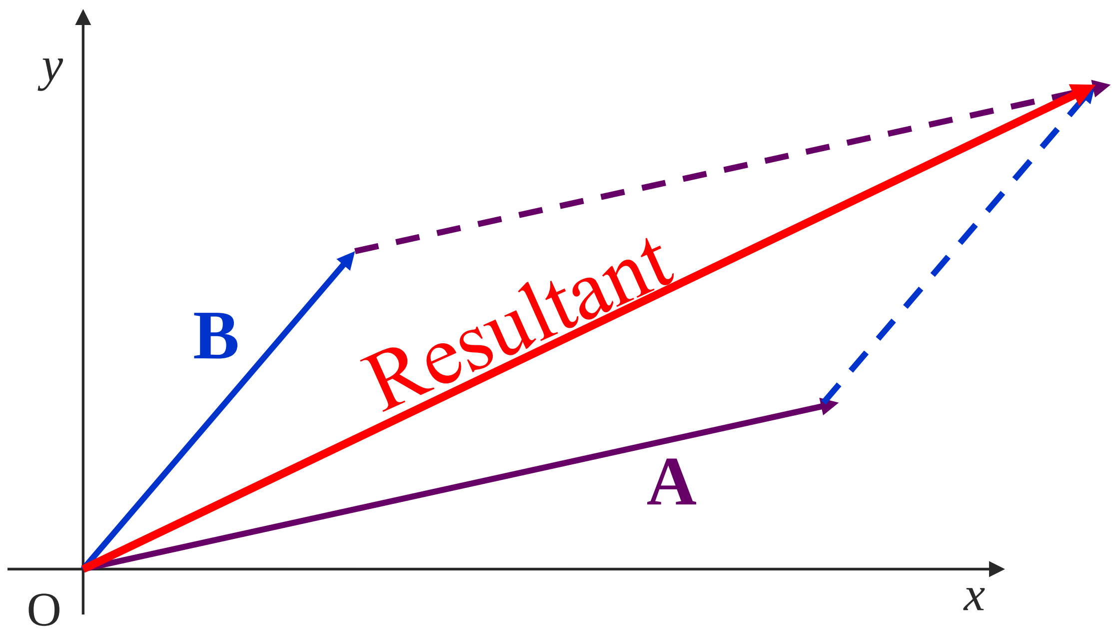
Vector Addition

- The Tip-to-tail Method: **A** + **B** = **R**



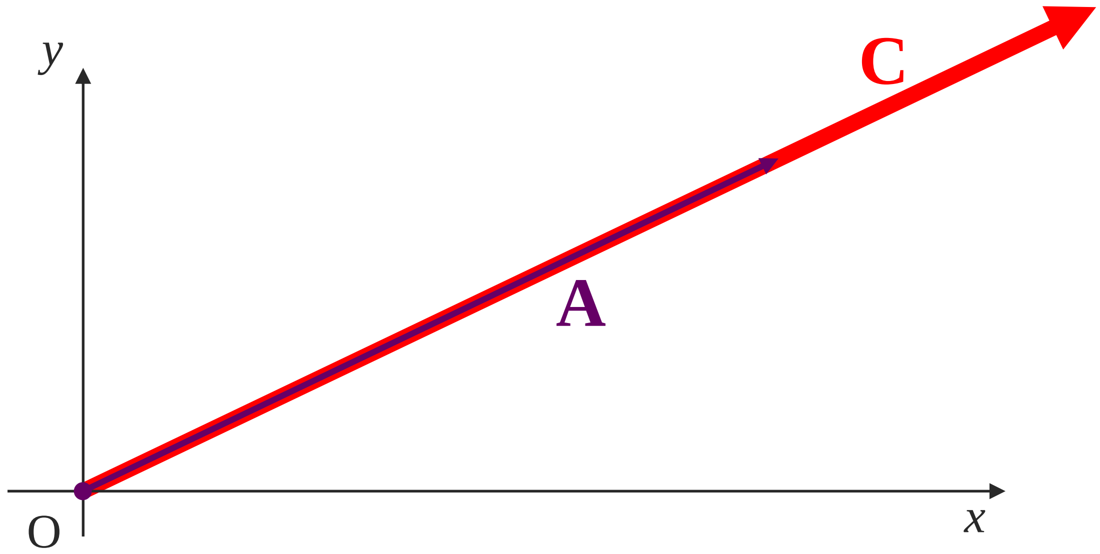
Vector Addition

- The Parallelogram Method: **A** + **B** = **R**

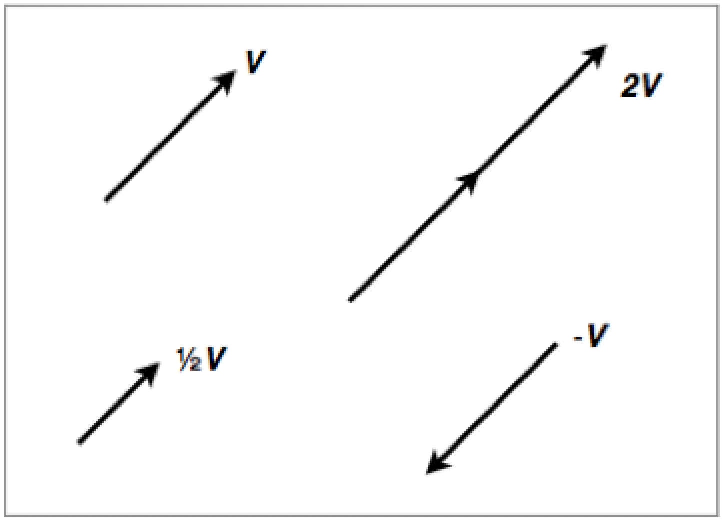


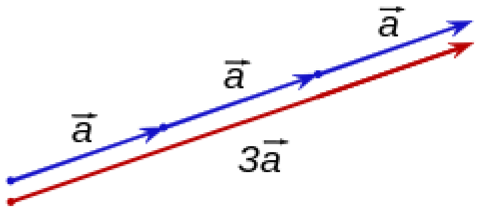
Scalar Multiple of Vector

- If vector **C** is in the ***same direction*** as vector **A** and **C** has a magnitude ***n*** times that of **A**, then **C** = ***n*A**.



Examples



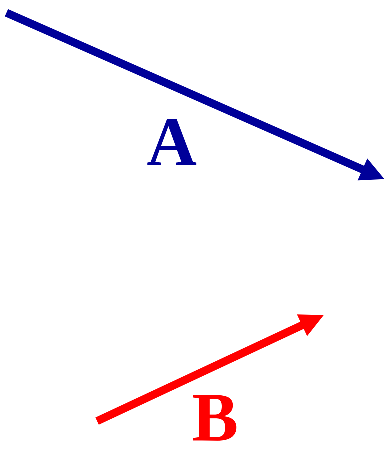


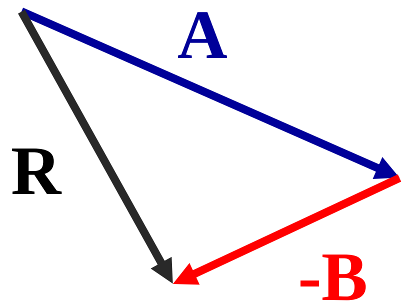
Subtraction of Vectors

- To subtract vectors, the direction of the vector is reversed.
- Therefore, **A** – **B** = **A** + (**-B**), where the minus sign indicates that vector **-B** has the ***opposite direction*** of vector **B**.

Subtraction of Vectors

- An illustration: **A** – **B** = **A** + (- **B**) = **R**

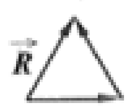


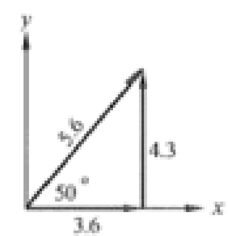


Questions EX 9.1   9-12, 15-17,

15. 5.6 cm, 50°

9.





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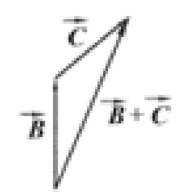

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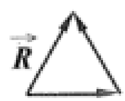

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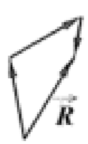

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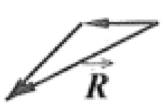
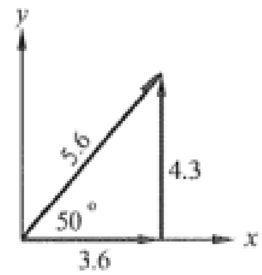
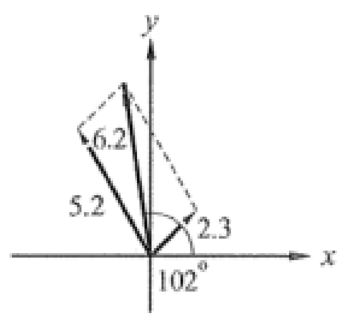


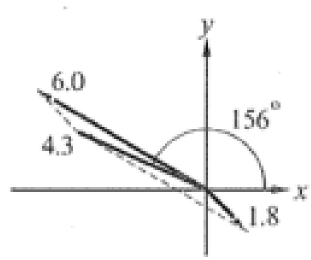
15. 5.6 cm, 50°



16. 6.2 cm, 102°



17. 4.3 cm, 156°



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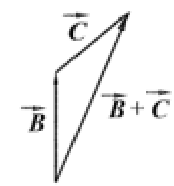
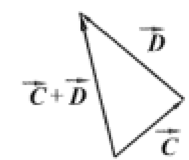


21.



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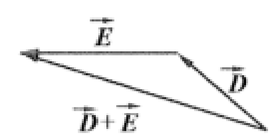
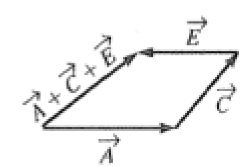


23.

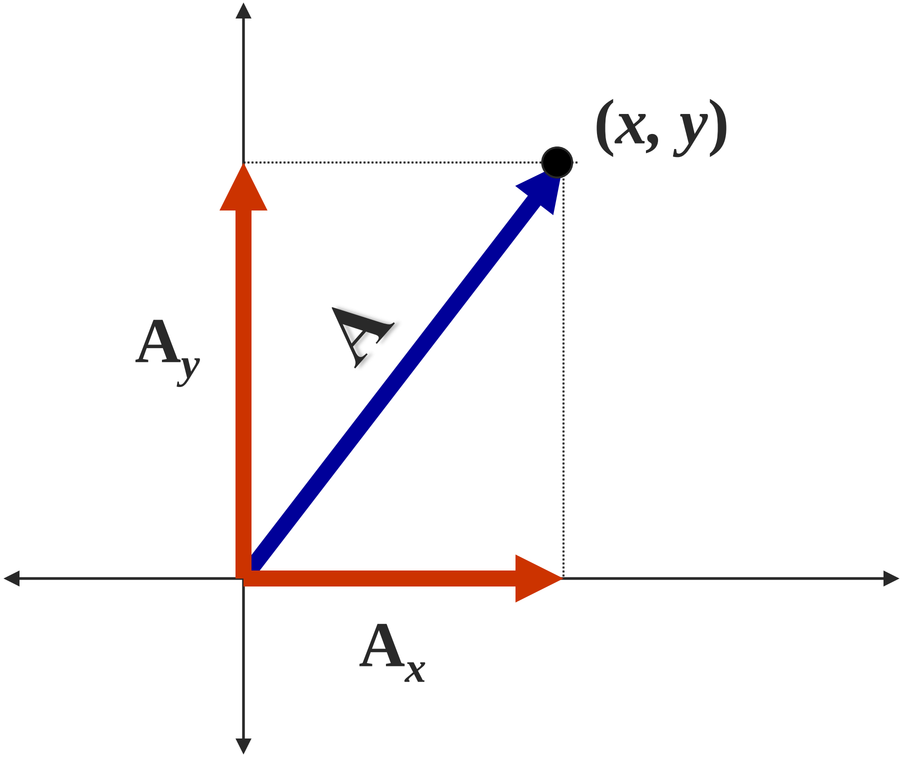


Ch. 9.2: Components of Vectors

- Two vectors that, when added together, have a resultant equal to the original vector, are called ***components***.
- Any vector can be replaced by its ***x-*** & ***y-*** components.
- Finding these component vectors is called ***resolving*** the vector.

Vector Components

- Components of a vector:



Resolving Vectors

- We will use trigonometry in our calculations by means of:

$$A_x = r * \cos \theta$$

$$A_y = r * \sin \theta$$

Steps Used in Finding the x- and y-Components of a Vector

1. Place vector **A** such that θ is in standard position.
2. Calculate $\mathbf{A}_x$, and $\mathbf{A}_y$ from $\mathbf{A}_x = \mathbf{r}\cos\theta$ and $\mathbf{A}_y = \mathbf{r}\sin\theta$. We may use the reference angle if we note the direction of the component.
3. Check the components to see if each is in the correct direction and has a magnitude that is proper for the reference angle.

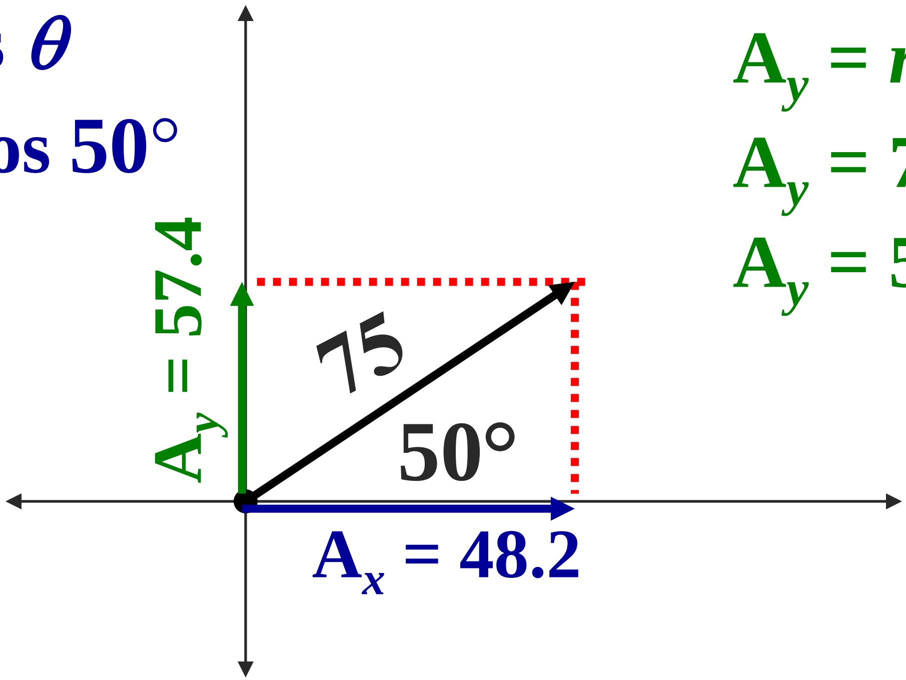
Example

- What are the rectangular components of the vector given magnitude of 75 & $\theta = 50°$?

$A_x = r * \cos\theta$

$A_x = 75 * \cos 50°$

$A_x = 48.2$



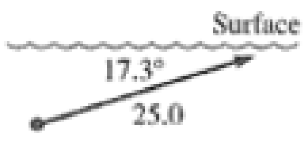
$A_y = r * \sin\theta$

$A_y = 75 * \sin 50°$

$A_y = 57.4$

Questions EX 9.2   5-8, 21-22,

21.



5. $V_x = 750\cos 28^\circ = 662$

$V_y = 750\sin 28^\circ = 352$

$V_x = 25.0\cos 17.3^\circ = 23.9$ km/h

$V_y = 25.0\sin 17.3^\circ = 7.43$ km/h

A

5. $V_x = 750\cos 28^\circ = 662$

$V_y = 750\sin 28^\circ = 352$

6. $V_x = 750\cos 105^\circ = -750\cos 75^\circ = -194$

$V_y = 750\sin 105^\circ = 750\sin 75^\circ = 724$

7. $V_x = -750\cos 62.3^\circ = 750\cos 242.3^\circ = -349$

$V_y = -750\sin 62.3^\circ = 750\sin 242.3^\circ = -664$

8. $V_x = 750\cos 52.4^\circ = 750\cos 307.6^\circ = 458$

$V_y = -750\sin 52.4^\circ = 750\sin 307.6^\circ = -594$

21.

$V_x = 25.0\cos 17.3^\circ = 23.9$ km/h

$V_y = 25.0\sin 17.3^\circ = 7.43$ km/h

22. $V_x = V\cos 66.4^\circ = 18.0(0.400) = 7.21$ ft/s

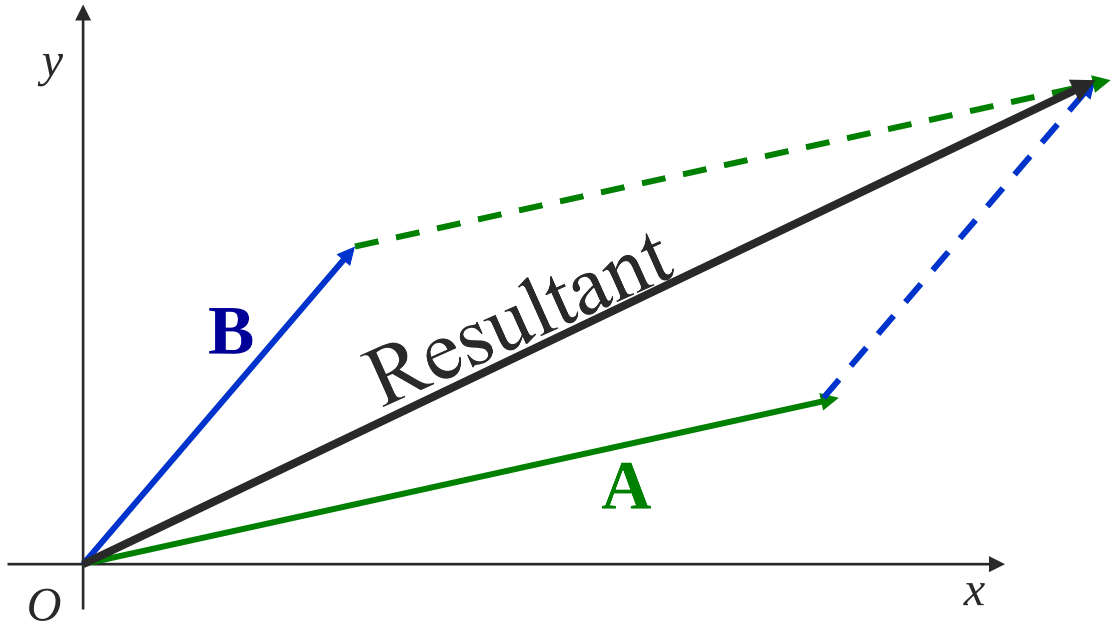
$V_y = V\sin 66.4^\circ = -18.0(0.916) = -16.5$ ft/s

Ch. 9.3: Vector Addition by Components

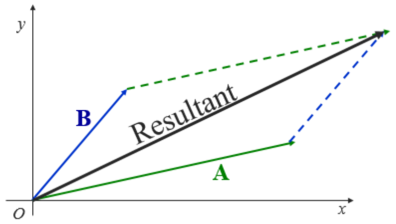
- To add two vectors:
  1. Place each vector with its tail at the origin.
  2. Resolve each vector into its ***x***- and ***y***-components
  3. Add the ***x***-components of each vector together.
  4. Add the ***y***-components of each vector together.
  5. Using the Pythagorean Theorem, find the magnitude of the resultant vector.
  6. Using the tangent ratio, find the direction of the resultant vector.

Example

- Vector **A**: 38.6∠16.6°
- Vector **B**: 28.3∠58°



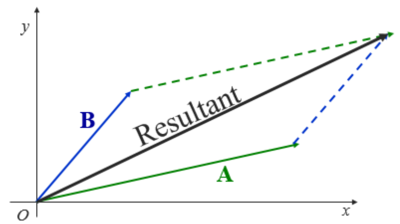
Solution – *finding the x-component*



- Vector **A**: **38.6∠16.6°**
- Vector **B**: **28.3∠58°**
- $\mathbf{A}x + \mathbf{B}x = \mathbf{R}x$
- $\mathbf{A}x = r\cos\theta = 38.6\cos 16.6° =$ **37**
- $\mathrm{B}x = r\cos\theta = 28.3\cos 58° =$ **15**

Rx = 52

Solution – *finding the y-component*

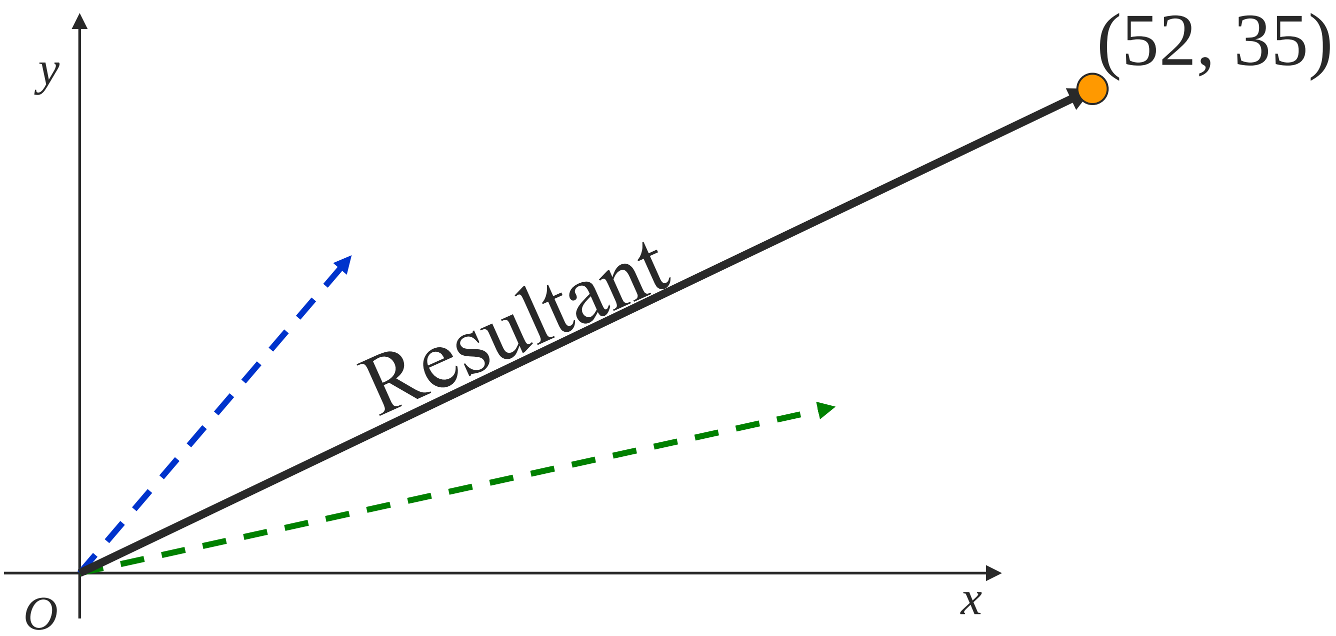


- Vector **A**: **38.6∠16.6°**
- Vector **B**: **28.3∠58°**
- $A_y + B_y = R_y$
- $A_y = r\sin\theta = 38.6\sin 16.6° =$ **11**
- $B_y = r\sin\theta = 28.3\sin 58° =$ **24**

Ry = 35

Solution (*continued*)

- Resultant Vector: (52, 35)



Solution (*continued*)

- Resultant Vector: (52, 35)
- What is the magnitude & direction of this resultant vector?

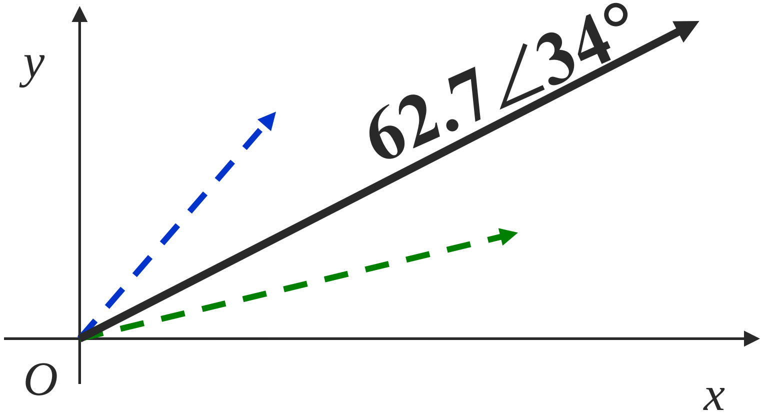
$$r = \sqrt{x^2 + y^2} = \sqrt{\left(52^2 + 35^2\right)} = 62.7$$

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(35/52) = 34°$$

- Answer: 62.7, ∠34°

Solution (*continued*)

- Resultant Vector:
- In rectangular form: (52, 35)
- Magnitude & direction: 62.7, ∠34°



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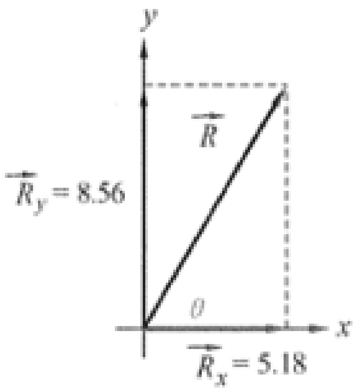
Ex 9.3 questions 7 – 10

Plus 27

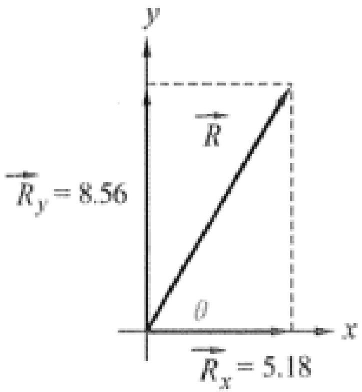
7.



$$R = \sqrt{5.18^2 + 8.56^2} = 10.0$$

$$\theta = \tan^{-1}\frac{8.56}{5.18} = 58.8°$$

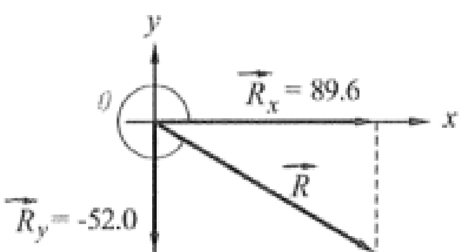
7.



$R = \sqrt{5.18^2 + 8.56^2} = 10.0$

$\theta = \tan^{-1}\dfrac{8.56}{5.18} = 58.8°$

8.



$R_x = 89.6, R_y = -52.0$

$R = \sqrt{R_x^2 + R_y^2} = \sqrt{89.6^2 + (-52.0)^2} = 104$

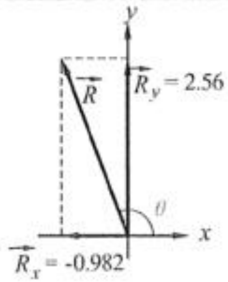
$\tan\theta_{ref} = \left|\dfrac{-52.0}{89.6}\right| = 0.580$

$\theta_{ref} = 30.1°$

$\theta = 360° - 30.1° = 329.9°$

(θ is in Quad IV since R_x is positive and R_y is negative)

9.



$R_x = -0.982, R_y = 2.56$

$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-0.982)^2 + 2.56^2} = 2.74$

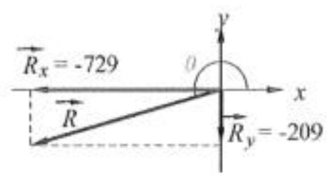
$\tan\theta_{ref} = \left|\dfrac{2.56}{-0.982}\right| = 2.61$

$\theta_{ref} = 69.0°$

$\theta = 180° - 69.0° = 111.0°$

(θ is in Quad II since R_x is negative and R_y is positive)

10.



$R_x = -729, R_y = -209$

$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-729)^2 + (-209)^2}$

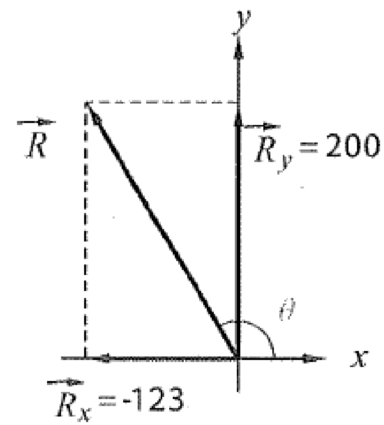
$\tan\theta_{ref} = \left|\dfrac{-209}{-729}\right| = 0.287$

$\theta_{ref} = 16.0°$

$\theta = 180° + 16.0° = 196.0°$

(θ is in Quad III since R_x is negative and R_y is negative)

27.



$R_x = 302\cos(180° - 45.4°) + 155\cos(180° + 53.0°)$
$+ 212\cos 30.8° = -123$

$R_y = 302\sin(180° - 45.4°) + 155\sin(180° + 53.0°)$
$+ 212\sin 30.8° = 200$

$R = \sqrt{R_x^2 + R_y^2} = 235$

$\theta = \tan^{-1}\dfrac{R_y}{R_x} = -58.4°$

$= 180° - 58.4° = 121.6°$

Ch. 9.4: Application of vectors

■ An aircraft's *heading* is the direction in which it is pointed. Its *air speed* is the speed at which it travels through the air surrounding it. Due to the wind, the heading and air speed, and its actual direction and speed relative to the ground, will differ.

EXAMPLE 3 Airplane velocity

An airplane headed due east is in a wind blowing from the southeast. What is the resultant velocity of the plane with respect to the surface of the earth if the velocity of the plane with respect to the air is 600 km/h and that of the wind is 100 km/h? See Fig. 9.35.

Let $\mathbf{v}_{px}$ be the velocity of the plane in the x-direction (east), $\mathbf{v}_{py}$ the velocity of the plane in the y-direction, $\mathbf{v}_{wx}$ the x-component of the velocity of the wind, $\mathbf{v}_{wy}$ the y-component of the velocity of the wind, and $\mathbf{v}_{pa}$ the velocity of the plane with respect to the air. Therefore,

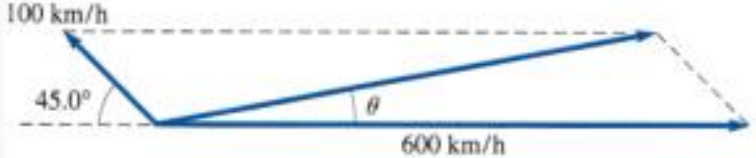


Fig. 9.35

$$v_{px} = v_{pa} - v_{wx} = 600 - 100(\cos 45.0°) = 529 \text{ km/h}$$

$$v_{py} = v_{wy} = 100(\sin 45.0°) = 70.7 \text{ km/h}$$

$$v = \sqrt{(529)^2 + (70.7)^2} = 534 \text{ km/h}$$

$$\theta = \tan^{-1}\frac{v_{py}}{v_{px}} = \tan^{-1}\frac{70.7}{529}$$

$$= 7.6°$$

The plane is traveling 534 km/h and is flying in a direction 7.6° north of east. From this, we observe that a plane does not necessarily head in the direction of its destination. ■

Equilibrium of Forces

As we have seen, an important vector quantity is the force acting on an object. One of the most important applications of vectors involves forces that are in **equilibrium.**

NOTE ▸ *For an object to be in equilibrium, the net force acting on it in any direction must be zero.* This condition is satisfied if the sum of the x-components of the force is zero and the sum of the y-components of the force is also zero. The following two examples illustrate forces in equilibrium.

EXAMPLE 4 Forces on block on inclined plane

A cement block is resting on a straight inclined plank that makes an angle of 30.0° with the horizontal. If the block weighs 80.0 lb, what is the force of friction between the block and the plank?

The weight of the cement block is the force exerted on the block due to gravity. Therefore, the weight is directed vertically downward. The frictional force tends to oppose the motion of the block and is directed upward along the plank. The frictional force must be sufficient to counterbalance that component of the weight of the block that is directed down the plank for the block to be at rest (not moving). The plank itself "holds up" that component of the weight that is perpendicular to the plank. A convenient set of coordinates (see Fig. 9.36) is one with the origin at the center of the block and with the x-axis directed up the plank and the y-axis perpendicular to the plank. The magnitude of the frictional force $\mathbf{F}_f$ is given by

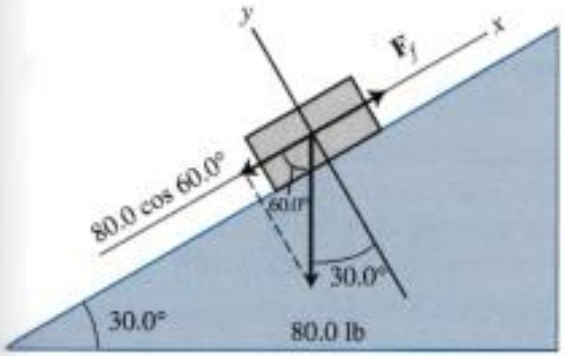


Fig. 9.36

$$F_f = 80.0 \cos 60.0^\circ$$

component of weight down plank equals frictional force

$$= 40.0 \text{ lb}$$

We have used the 60.0° angle since it is the reference angle. We could have expressed the frictional force as $F_f = 80.0 \sin 30.0^\circ$.

Here, it is assumed that the block is small enough that we may calculate all forces as though they act at the center of the block (although we know that the frictional force acts at the surface between the block and the plank). ■

■ Newton's *third law of motion* states that when an object exerts a force on another object, the second object exerts on the first object a force of the same magnitude but in the opposite direction. The force exerted by the plank on the block is an illustration of this law. (Sir Isaac Newton, again. See page 268.)

Ch. 9.5: Oblique Triangles, the Law of Sines

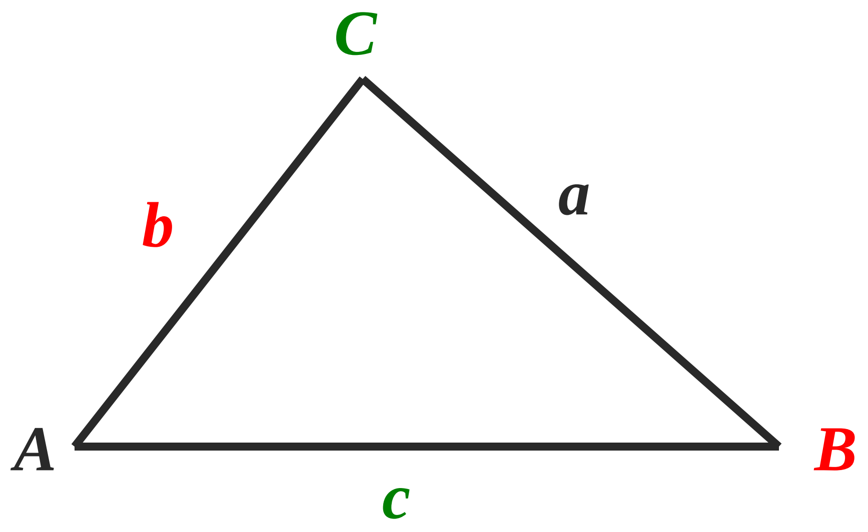
- We will use the Law of Sines & Cosines when finding the angles & sides of a triangle which is ***not*** a right triangle.
- This type of triangle is known as an ***oblique*** triangle.

Possible Combinations of Oblique Triangles to Solve

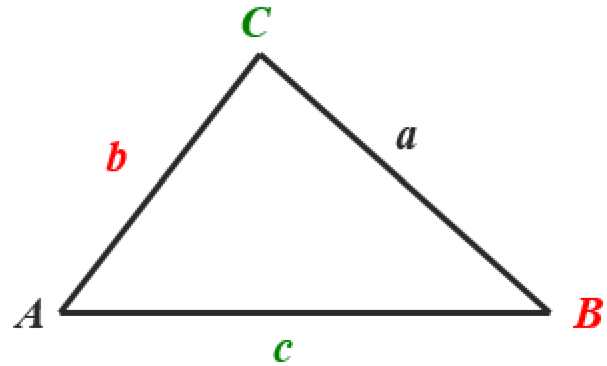
- ***Case* 1**. Two angles and one side
- ***Case* 2**. Two sides and the angle opposite one of them
- ***Case* 3**. Two sides and the included angle
- ***Case* 4**. Three sides

Standard Notation for Triangles

- We will use standard notation in our discussion of all triangles.



Law of Sines

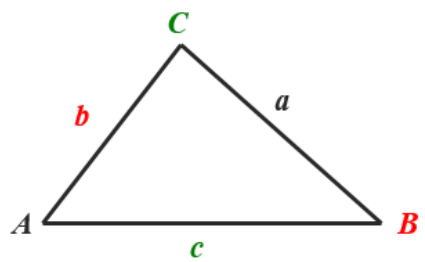


- If **ABC** is a triangle with sides ***a***, ***b*** and ***c***, then the sides of the triangle are proportional to the ***sines*** of the opposite angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Case 1. Two Angles and One Side

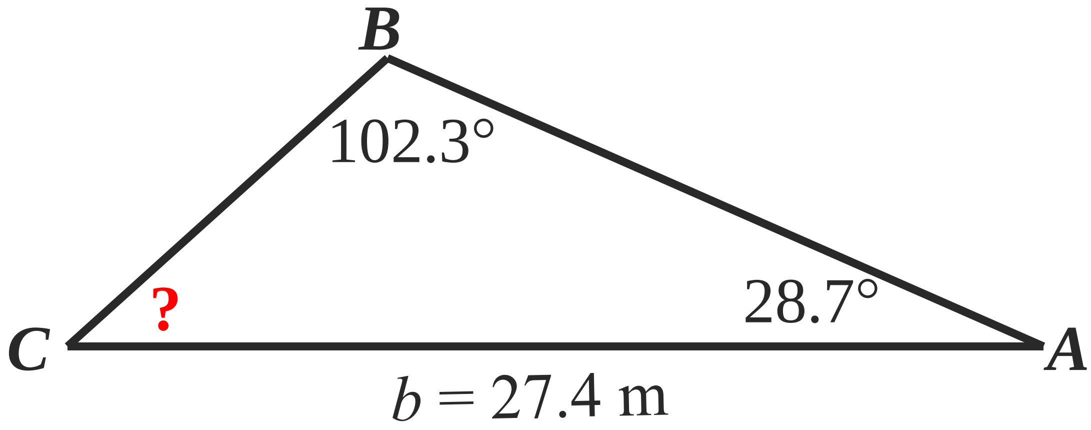
- Given $A = 28.7°$, $B = 102.3°$, $b = 27.4$ m, find a, c, and C.
- ***Solution***:



1. Make a sketch.
2. Use the triangle sum theorem to find the missing angle, C.
3. Use the Law of Sines to find the missing lengths.

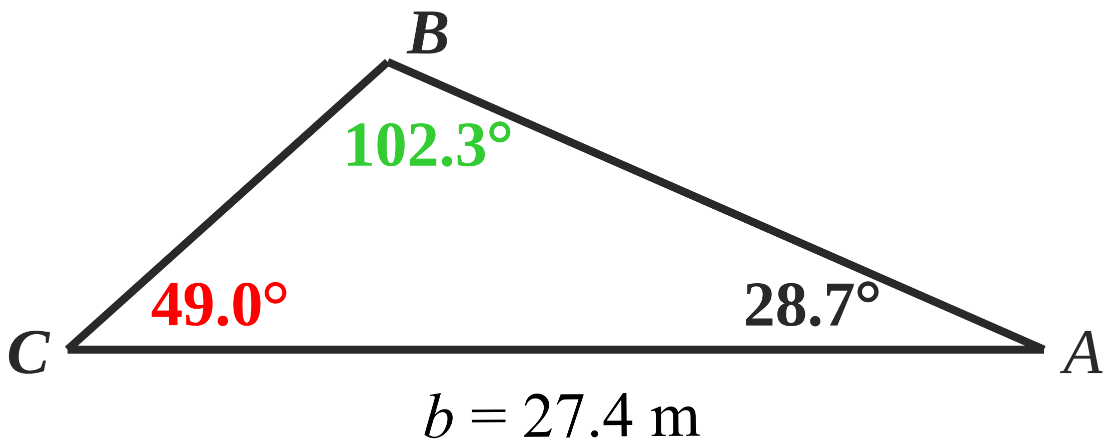
Solution (*continued*)

Making a sketch reveals …



Find the missing angle *C* …

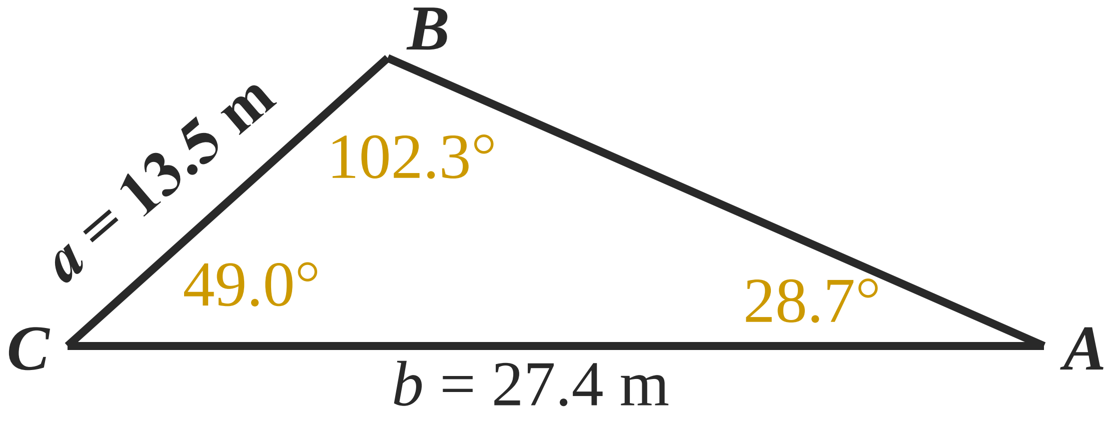
Solution (*continued*)



The missing angle *C* is:

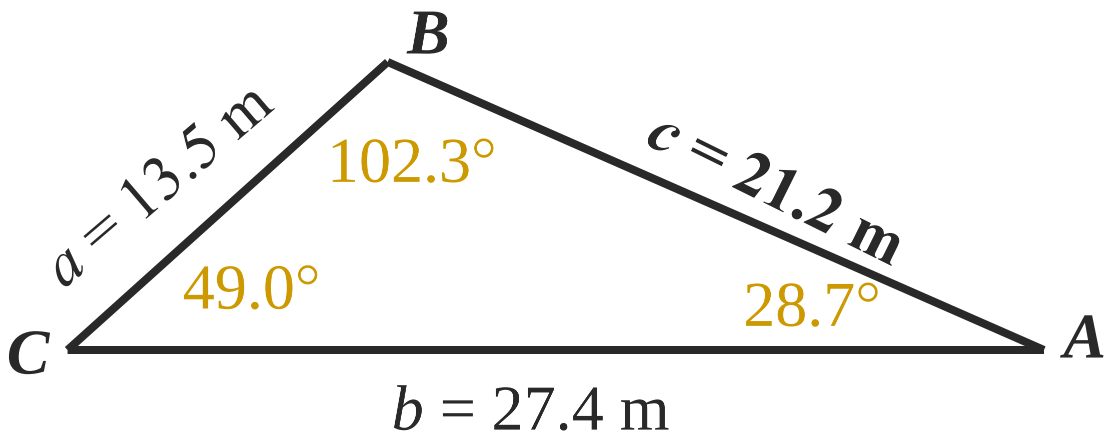
$$180° - 28.7° - 102.3° = 49°$$

Solution (*continued*)



Length *a* is: $$\frac{a}{\sin 28.7°} = \frac{27.4}{\sin 102.3°}$$

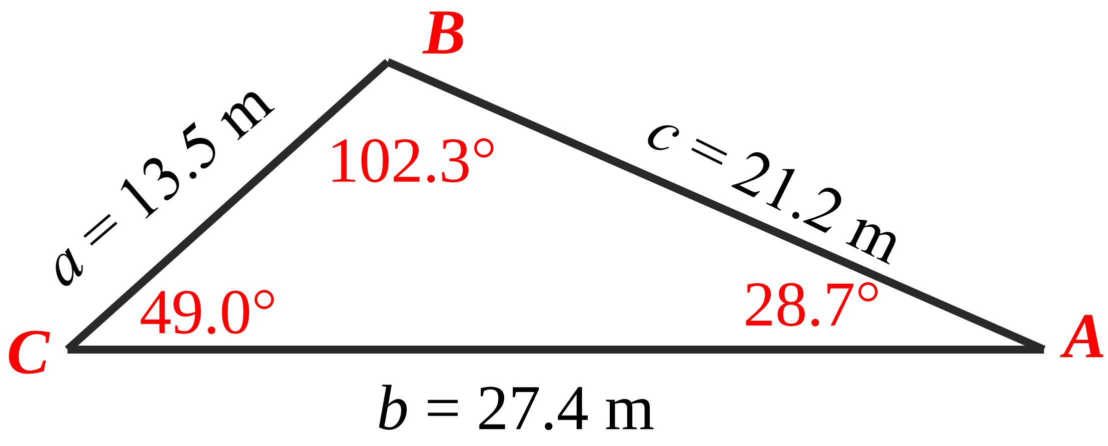
Solution (*continued*)



Length *c* is:

$$\frac{c}{\sin 49°} = \frac{27.4}{\sin 102.3°}$$

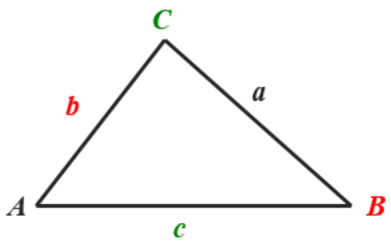
Final Solution



To have ***solved*** this triangle, the values of all 3 lengths have been found and the size of all 3 angles have been found.

Case 2. Two Sides and the Angle Opposite One of Them

- Given $A = 35°$, $a = 20$, $b = 30$ m, find c, B, and C.



- ***Solution***:
  1. Make a sketch.
  2. Use the Law of Sines to find B.
  3. Use the Triangle Sum theorem to find C.
  4. Use the Law of Sines to find length c.

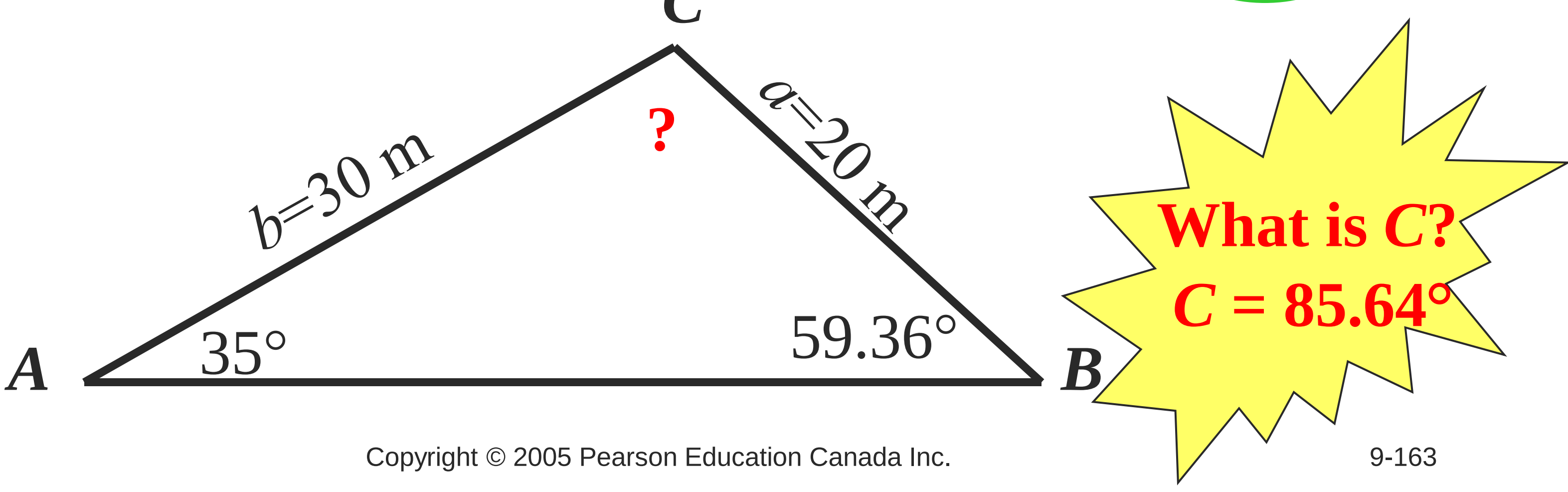
Solution

- Solving using the Sine Law gives us:

$$\frac{20}{\sin 35^\circ} = \frac{30}{\sin B} \rightarrow 20\sin B = 30\sin 35^\circ$$

$$\sin B = 0.8604$$

$$B = \sin^{-1} 0.8604 = 59.36^\circ$$

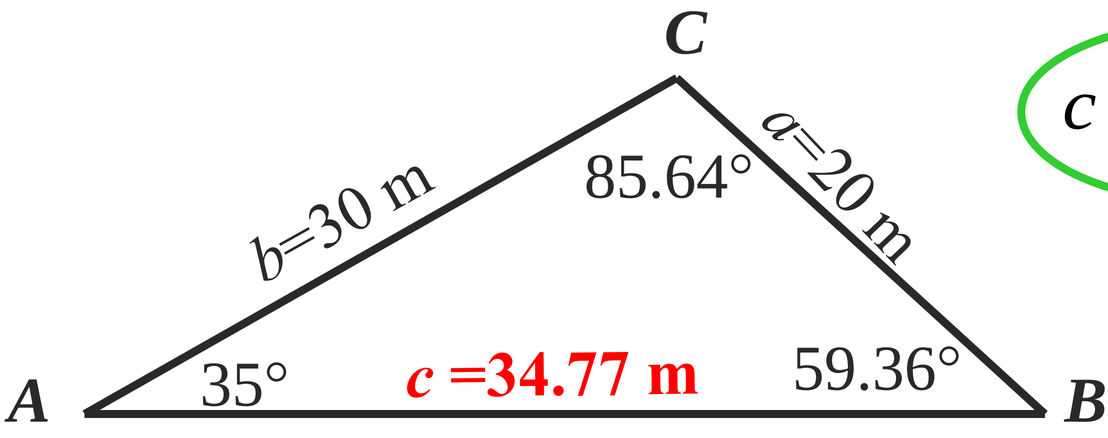


Solution (*continued*)

- Use the Sine Law to find length c:

$$\frac{20}{\sin 35°} = \frac{c}{\sin 85.64°} \rightarrow c \sin 35° = 20 \sin 85.64°$$

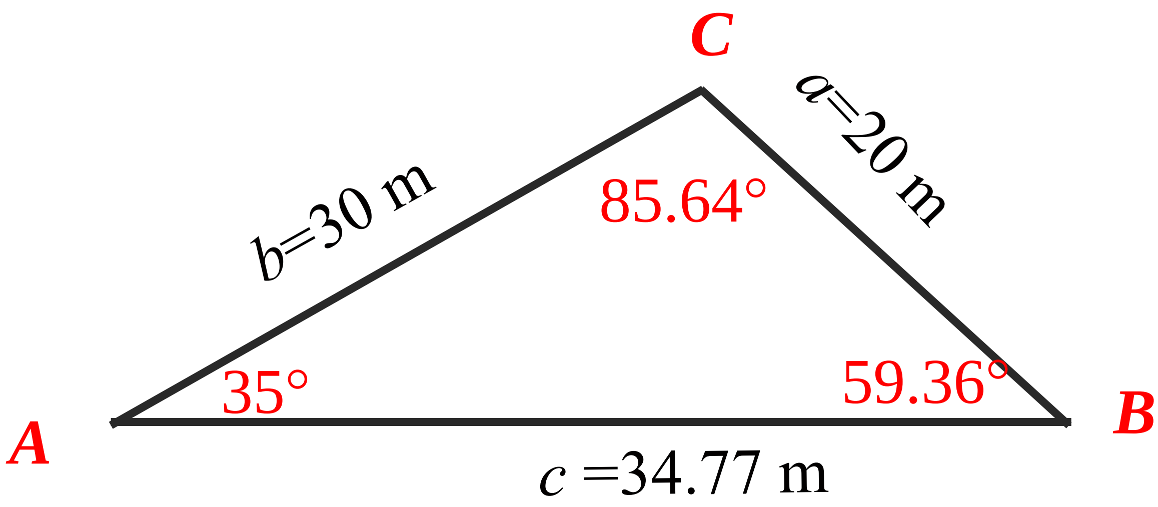
$$c = \frac{19.942}{0.5736}$$



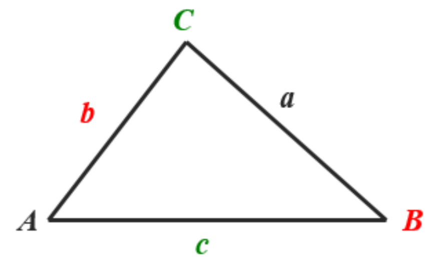
$$c = 34.77 m$$

Solution (*continued*)

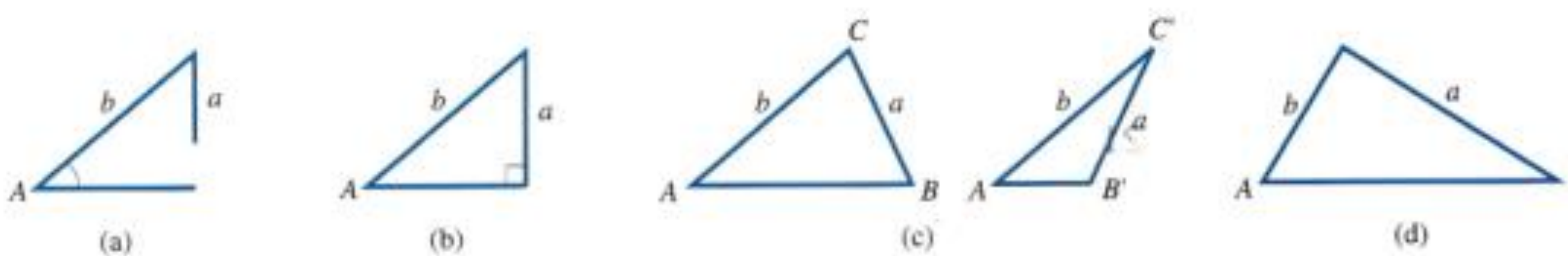
To have ***solved*** this triangle, the values of all 3 lengths have been found and the size of all 3 angles have been found.



Summary of Solutions:



- Two Sides and the Angle Opposite One of Them
  1. No solution if $a < b sin(\theta)$.
  2. A right triangle solution if $a = b \sin A$.
  3. Two solutions if $b \sin A < a < b$.
  4. One solution if $a > b$.



Summary of Solutions:

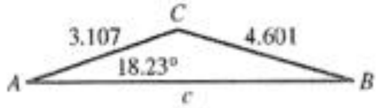
- To have 2 solutions, we must know 2 sides and the angle opposite one of the sides, and the shorter side must be opposite the known angle.
- If there is no solution, the calculator will indicate an error.
- If the solution is a right triangle, the calculator will show an angle of exactly 90˚.
- Since 2 solutions may result from Case 2, it is often called the ***ambiguous case***.

?

Exercise 9.5 questions 7-10, 28,29

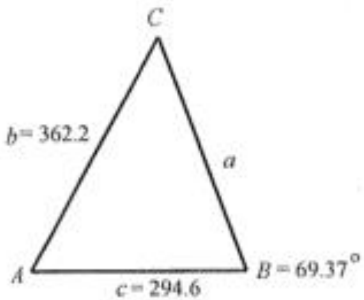
A

7.



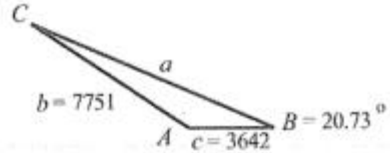
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{C}{\sin C}$$
$$= \frac{4.601}{\sin 18.23°} = \frac{3.107}{\sin B} = \frac{C}{\sin C}$$
$$\sin B = \frac{3.107 \sin 18.23°}{4.601}$$
$$B = 12.20°,\ C = 180 - 18.23 - 12.20$$
$$= 149.57°$$
$$c = \frac{4.601 \sin 149.57°}{\sin 18.23°} = 7.448$$

8.



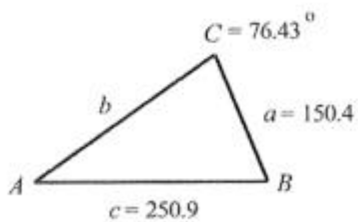
$$b = 362.2,\ c = 294.6,\ B = 69.37°$$
$$\frac{b}{\sin B} = \frac{c}{\sin C};\ \frac{362.2}{\sin 69.37°} = \frac{294.6}{\sin C}$$
$$\sin C = \frac{294.6 \sin 69.37°}{362.2}$$
$$C = 49.57°$$
$$A = 180.0° - (69.37° + 49.57°) = 61.06°$$
$$\frac{a}{\sin A} = \frac{b}{\sin B};\ \frac{a}{\sin 61.06°} = \frac{362.2}{\sin 69.37°}$$

9.



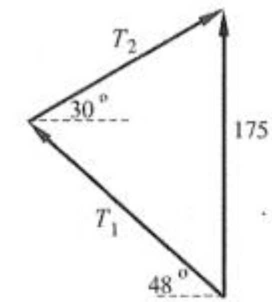
$$b = 7751,\ c = 3642,\ B = 20.73°$$
$$\frac{b}{\sin B} = \frac{c}{\sin C};\ \frac{7751}{\sin 20.73°} = \frac{3642}{\sin C}$$
$$\sin C = \frac{3642 \sin 20.73°}{7751} = 0.1663$$
$$C = 9.57°$$
$$A = 180.0° - (20.73° - 9.57°) = 149.70°$$
$$\frac{a}{\sin A} = \frac{b}{\sin B};\ \frac{a}{\sin 149.70°} = \frac{7751}{\sin 20.73°}$$
$$a = \frac{7751 \sin 149.70°}{\sin 20.73°} = 11,050$$

10.



$$a = 150.4,\ c = 250.9,\ C = 76.43°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C};\ \frac{150.4}{\sin A} = \frac{250.9}{\sin 76.43°}$$
$$\sin A = \frac{150.4 \sin 76.43°}{250.9} = 0.5827$$
$$A = 35.64°$$
$$B = 180.0° - (35.64° - 76.43°) = 67.93°$$
$$\frac{b}{\sin B} = \frac{c}{\sin C};\ \frac{b}{\sin 67.93°} = \frac{250.9}{\sin 76.43°}$$
$$b = \frac{250.9 \sin 67.93°}{\sin 76.43°} = 239.2$$

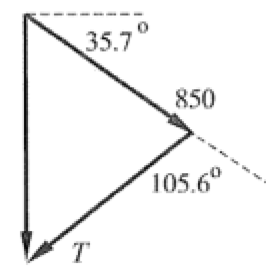
28.



175 is vertical

$$\frac{T_1}{\sin 60.0°} = \frac{175}{\sin 72.0°};\ T_1 = \frac{175 \sin 60.0°}{\sin 72.0°} = 159 \text{ lb}$$
$$\frac{T_2}{\sin 48.0°} = \frac{175}{\sin 72.0°};\ T_2 = \frac{175 \sin 48.0°}{\sin 72.0°} = 137 \text{ lb}$$

29.



$$\frac{T}{\sin 54.3°} = \frac{850}{\sin 51.3°};\ T = \frac{850 \sin 54.3°}{\sin 51.3°} = 880 \text{ N}$$

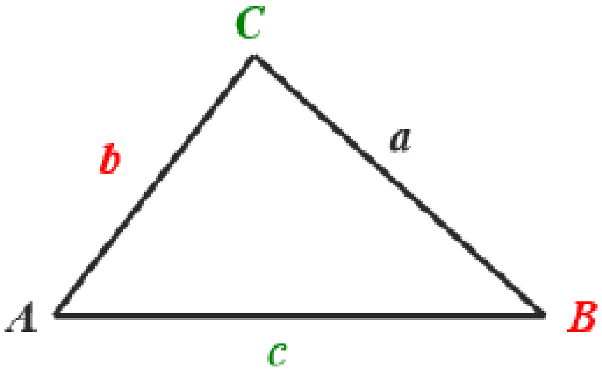
Ch. 9.6: The Law of Cosines

- The Law of Sines can only be used when we know:
  - ***Case* 3.** Two sides and the included angle
  - ***Case* 4**: Three known Sides.
- Use the Law of Cosine when all 3 sides are known & when there are 2 sides & an included angle.

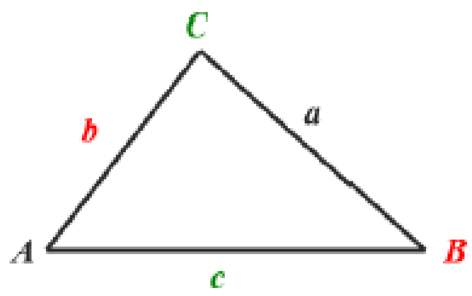
The Law of Cosines

- When we don't know $\angle B$ or $\angle C$, we use the formula:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Other Forms of the Law of Cosine



- When we don't know $\angle A$ or $\angle C$, we use the formula:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

- When we don't know $\angle A$ or $\angle B$, we use the formula:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

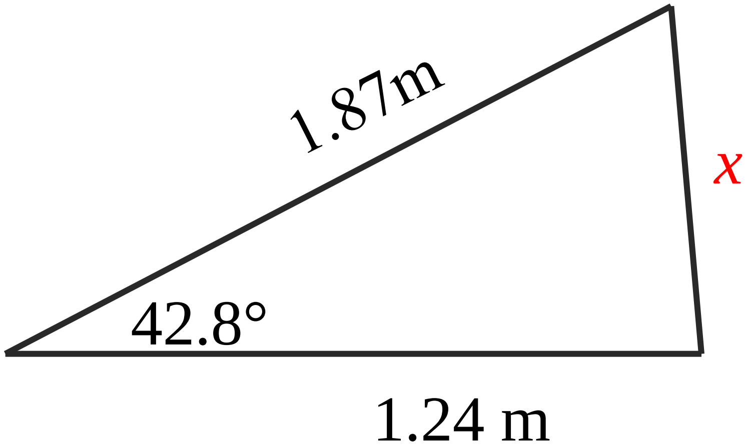
Case 3. Two sides and the included angle

- Given two lengths, 1.24 m and 1.87 m, and the contained angle 42.8°, what is the length of the third side?

First sketch the problem

Case 3. Two sides and the included angle

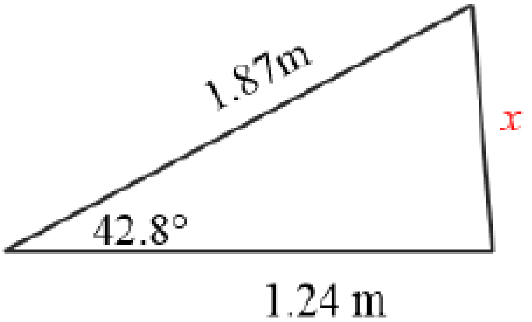
- Given two lengths, 1.24 m and 1.87 m, and the contained angle 42.8°, what is the length of the third side?



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Solution

- Given two lengths, 1.24 m and 1.87 m and the contained angle 42.8°, what is the length of the third side?
- Using the Cosine Law:



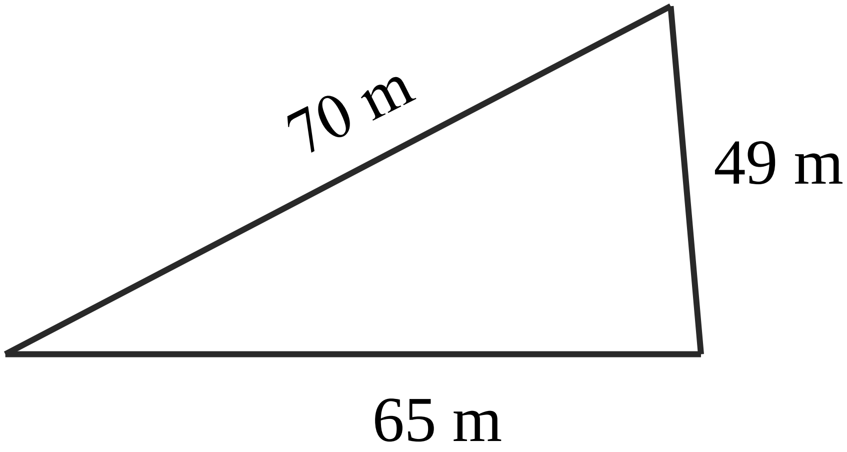
$$x^2 = 1.24^2 + 1.87^2 - 2(1.24)(1.87)\cos 42.8°$$

$$x = 1.28 \text{ m}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

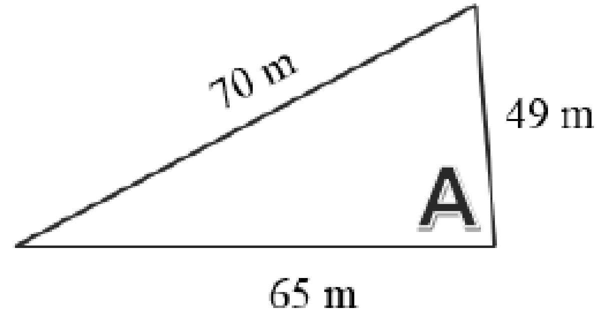
Case 4. Three Sides

- Given three lengths, 70 m and 65 m and 49 m, what are all 3 angles?



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Solution



- We use all 3 lengths in the Law of Cosines.
- Let $a = 70$ m, $b = 65$ m, $c = 49$ m

Find angle A:

$$70^2 = 65^2 + 49^2 - 2(65)(49)\cos .$$

$$A = 74.3°$$

Solution (*continued*)

Find angle B:

$$\frac{70}{\sin 74.3°} = \frac{65}{\sin B}$$

$$B = 63.4°$$

Find angle C:

$$C = 180° - 74.3° - 63.4°$$

Solving Oblique Triangles

- ***Case 1: Two Angles and One Side***
  - Find the unknown angle by subtracting the sum of the known angles from 180˚ (**the Triangle Sum Theorem**).
  - Use the Law of Sines to find the unknown sides.

Solving Oblique Triangles (*continued*)

- ***Case 2: Two Sides and the Angle Opposite One of Them***
  - Use the known side and the known angle opposite it to find the angle opposite the other known side.
  - Find the third angle from the fact that the sum of the angles is 180˚ (the Triangle Sum Theorem).
  - Use the Law of Sines to find the third side.
  - Watch for the ambiguous case.

Solving Oblique Triangles (*continued*)

- ***Case 3: Two Sides and the Included Angle***
  - Find the third side by using the Law of Cosines.
  - Find the smaller unknown angle (opposite the shorter side) by using the Law of Sines.
  - Complete the solution using the fact that the sum of the angles is180˚ (the Triangle Sum Theorem).

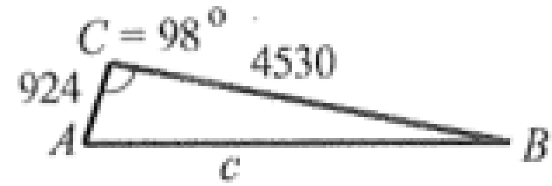
Solving Oblique Triangles (*continued*)

- ***Case 4: Three Sides***
  - Find the largest angle (opposite the longest side) by using the Law of Cosines. Find a second angle by using the Law of Sines.
  - Complete the solution by using the fact that the sum of the angles is 180˚ (the Triangle Sum Theorem).

?

Exercise 9.6 questions 5-9, 28, 31

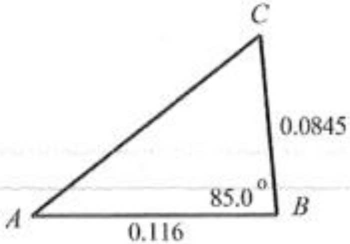
5.



$a = 4530, b = 924, C = 98.0^\circ$

c = 4750
B = 11.1°
A = 70.9°

6.



$a = 0.0845, c = 0.116, B = 85.0°$

$$b = \sqrt{0.0845^2 + 0.116^2 + 2(0.0845)(0.116)(\cos 85.0°)}$$
$$= 0.137$$

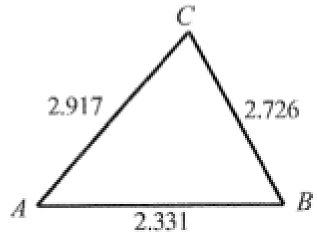
$$\frac{b}{\sin B} = \frac{a}{\sin A}; \frac{0.137}{\sin 85.0} = \frac{0.0845}{\sin A}$$

$$\sin A = \frac{0.0845 \sin 85.0°}{0.137} = 0.614$$

$$A = 37.9°$$

$$C = 180° - 37.9° - 85.0° = 57.1°$$

8.



$a = 2.331, b = 2.726, c = 2.917$

$$\cos A = \frac{2.726^2 + 2.917^2 - 2.331^2}{2(2.726)(2.917)} = 0.6606$$

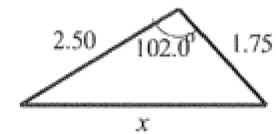
$$A = 48.65°$$

$$\cos B = \frac{2.331^2 + 2.917^2 - 2.726^2}{2(2.331)(2.917)} = 0.4788$$

$$B = 61.39°$$

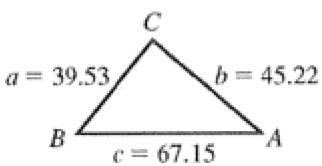
$$C = 180° - 48.65° - 61.39° = 69.96°$$

28.



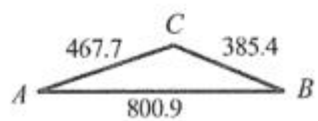
$$x = \sqrt{1.75^2 + 2.50^2 - 2(2.50)(1.75)(\cos 102.0°)}$$
$$= 3.34 \text{ m}$$

7.



$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$45.22^2 = 39.53^2 + 67.15^2 - 2(39.53)(67.15) \cos B$$
$$\cos B = 0.7585$$
$$B = 40.67°$$
$$b^2 + c^2 - 2bc \cos A = a^2$$
$$45.22^2 + 67.15^2 - 2(45.22)(67.15) \cos A = 39.53^2$$
$$\cos A = 0.8219$$
$$A = 34.73°$$
$$C = 180° - 40.67° - 34.73° = 104.6°$$

9.



$a = 385.4, b = 467.7, c = 800.9$

$$\cos A = \frac{467.7^2 + 800.9^2 - 385.4^2}{2(467.7)(800.9)} = 0.9499$$

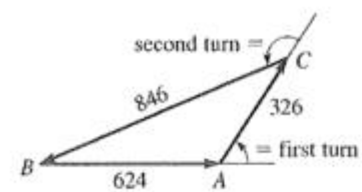
$$A = 18.21°$$

$$\cos B = \frac{385.4^2 + 800.9^2 - 467.7^2}{2(385.4)(800.9)} = 0.9253$$

$$B = 22.28°$$

$$C = 180° - 18.21° - 22.28° = 139.51°$$

31.



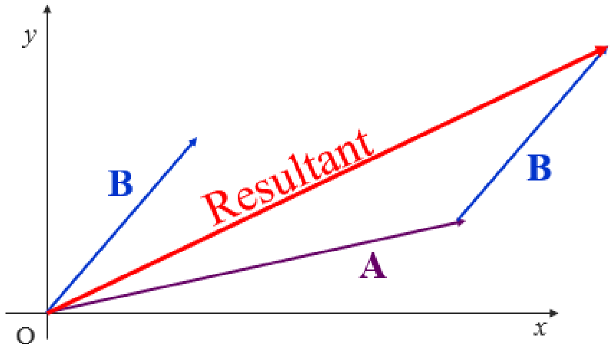
$$846^2 = 624^2 + 326^2 - 2(624)(326) \cos A$$
$$\cos A = -0.5409$$
$$A = 122.7°$$
$$\text{first turn} = 180° - 122.7° = 57.3°$$
$$624^2 = 846^2 + 326^2 - 2(846)(326) \cos C$$
$$\cos C = 0.7843$$
$$C = 38.3°$$
$$\text{second turn} = 180° - 38.3° = 141.7°$$

Application of vectors.

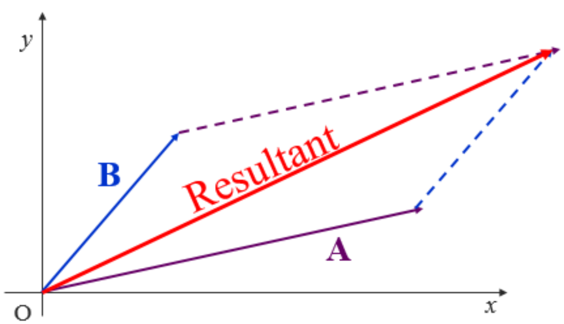
We can apply what we have learnt to real problems

Look back over chapter 9.4 and some examples

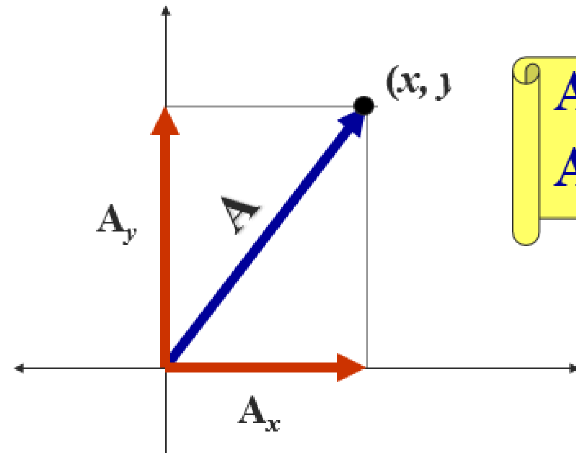
The Tip-to-tail Method: **A** + **B** = **R**



The Parallelogram Method: **A** + **B** = **R**



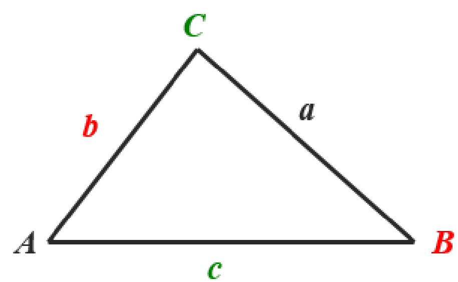
Components of a vector:



$$A_x = r^* \cos\ \theta$$
$$A_y = r^* \sin\ \theta$$

$$r = \sqrt{x^2 + y^2}$$

$|\mathbf{AB}|$ or AB



$$180^0 = A + B + C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

M2 Ch9

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Chapter 10

Graphing the Trigonometric Functions

Learning objectives

4) Diagram trigonometric functions.

I can………….

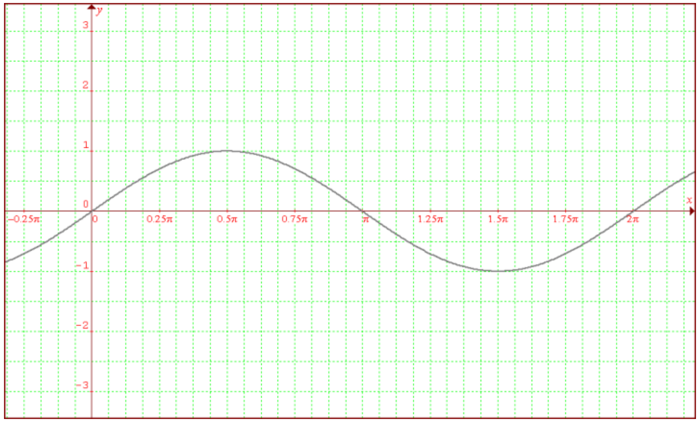
4.1 Sketch the curves of the trigonometric functions using amplitude, period and displacement.
4.2 Calculate the amplitude of a trigonometric function
4.3 Calculate the period of a trigonometric function
4.4 Calculate the displacement of the trigonometric function
4.5 From sketch calculate period.
4.6 From sketch calculate amplitude.
4.7 From sketch of a trigonometric function calculate displacement.
4.8 Sketch composed trigonometric functions (tangent, cotangent, secant, and cosecant).

Ch. 10.1: Graphs of $y = a\sin x$ & $y = a\cos x$

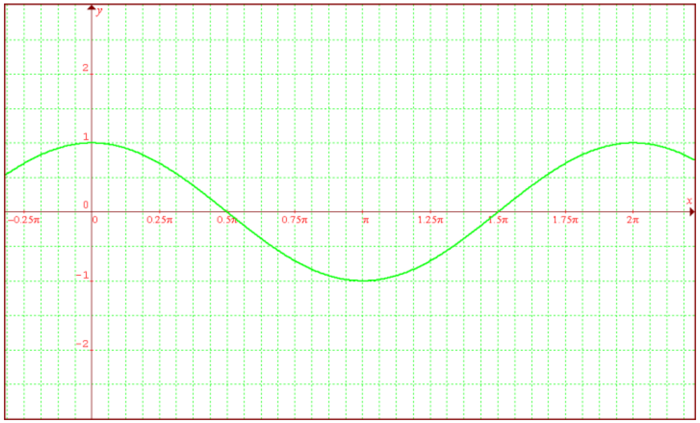
- The ***angles*** of the graphs of the trigonometric functions will be expressed in ***radians***.
- In this way, both the independent variable and the dependent variable are real numbers.

The Periodic Curve

- $y = \sin x$ and $y = \cos x$ are known as sinusoidal or periodic functions.



$y = \sin x$



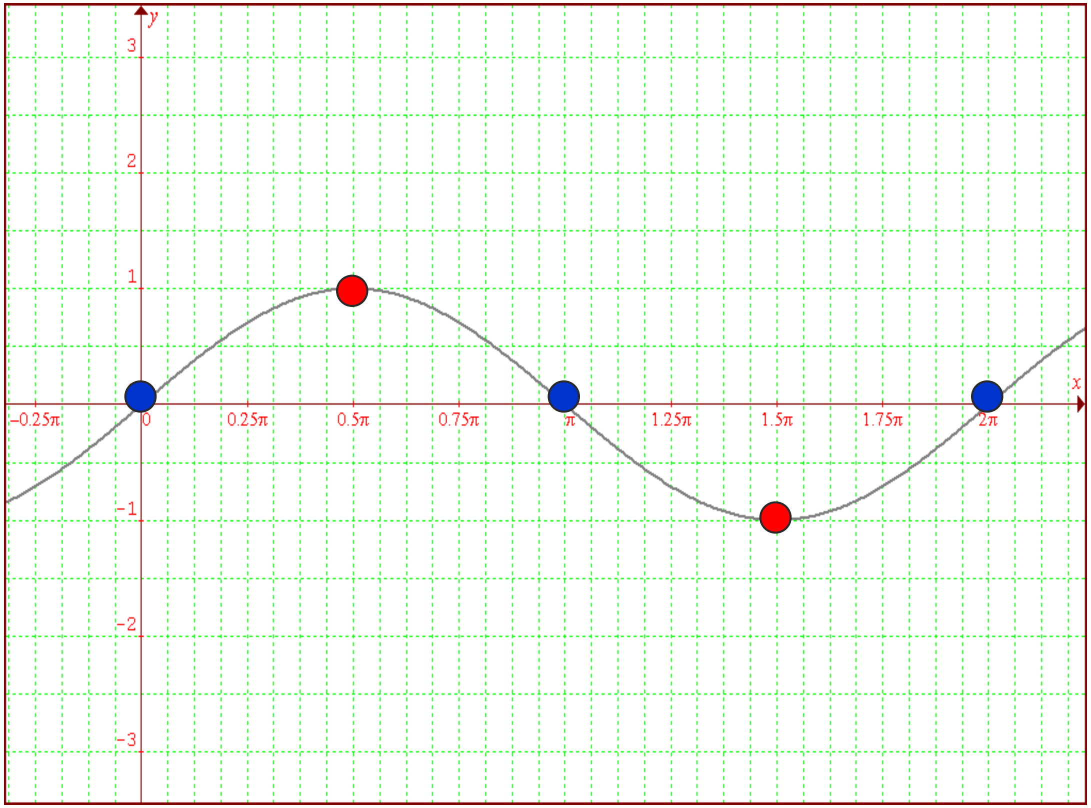
$y = \cos x$

Graphing the Sine & Cosine Curves

- There are 5 key points found within one cycle of the sine or cosine curve to use when sketching the trigonometric curves.
  - The intercepts (or zeros)
  - The maximum
  - The minimum
- These are important landmarks and will help you sketch a symmetrical curve.

Key Points on the Normal Sine Curve

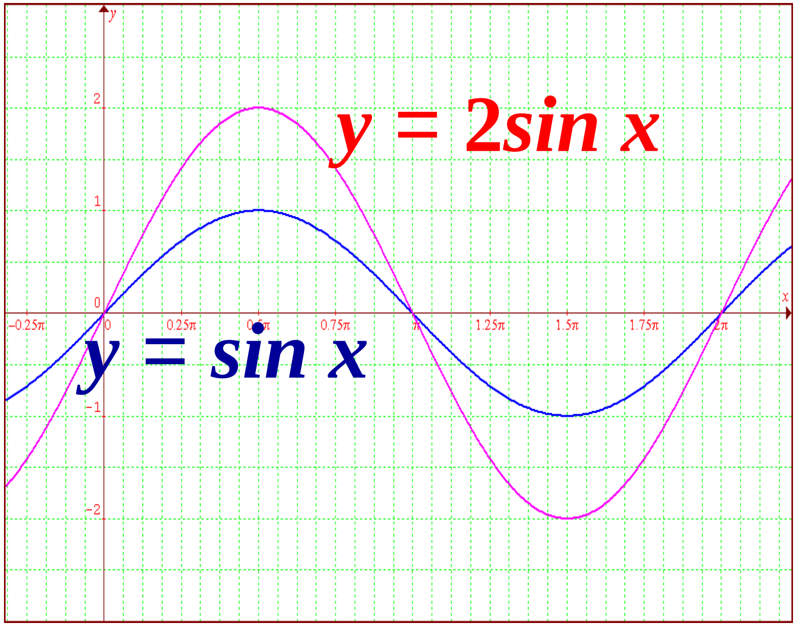
1. 1^{st} intercept: $(0, 0)$
2. 2^{nd} intercept: $(\pi, 0)$
3. 3^{rd} intercept: $(2\pi, 0)$
4. Maximum: $(\pi/2, 1)$
5. Minimum: $(3\pi/2, -1)$



Amplitude

- All y-values obtained for the graph of $y = \sin x$ and $y = \cos x$ are to be multiplied by a.
- The amplitude increases by $|a|$ amount.

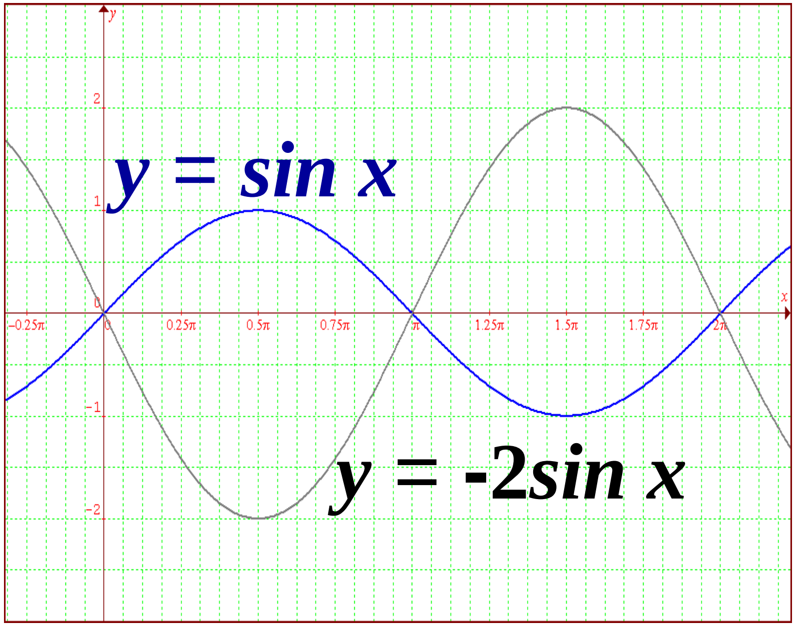
Amplitude



- Note the location of the 5 key points used in graphing.

Amplitude

- The negative reverses the direction of the graph.
- The intercepts remain the same but the maximum & minimum are reversed.

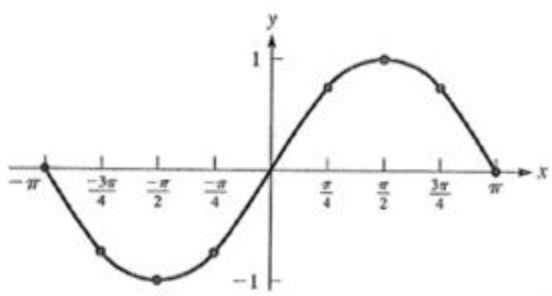


?

Ex 10.1 q 3 to 6

3. $y = \sin x$

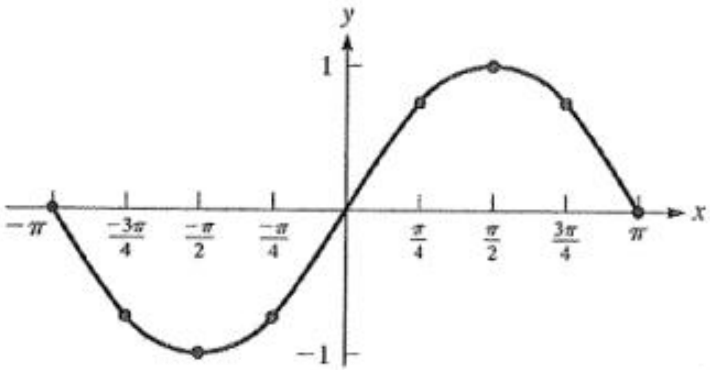
x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0



A

3. $y = \sin x$

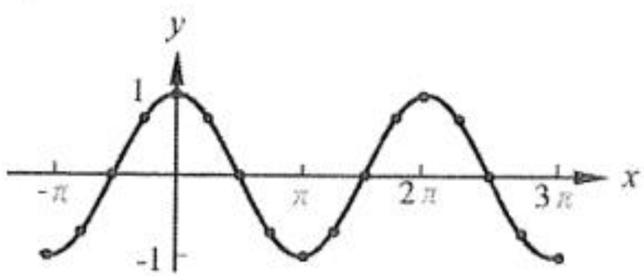
x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0



4. $y = \cos x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	-1	-0.7	0	0.7	1	0.7	0	-0.7	-1

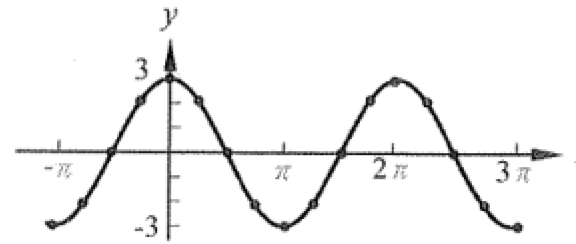
x	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-0.7	0	0.7	1	0.7	0	-0.7	-1



5. $y = 3\cos x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	-3	-2.1	0	2.1	3	2.1	0	-2.1	3

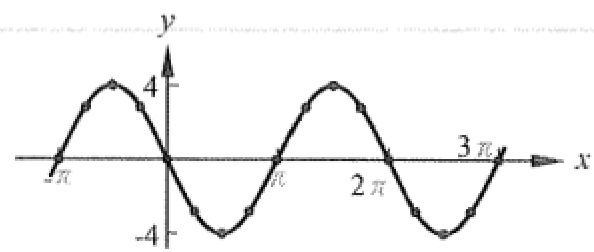
x	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-2.1	0	2.1	3	2.1	0	-2.1	-3



6. $y = -4\sin x$

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	2.8	4	2.8	0	-2.8	-4	-2.8	0

x	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	2.8	4	2.8	0	-2.8	-4	-2.8	0



Ch. 10.2: Graphs of $y = a\sin bx$ & $y = a\cos bx$

- When graphing $y = \sin x$ and $y = \cos x$ the values of y repeat every 2π radians.
- We say that these functions are periodic and have a period of 2π rads.
- Graphing $y = a\sin bx$ or $y = a\cos bx$
- **Period = $2\pi/b$**
- We still use the 5 key values to sketch a trigonometric function to include its period.

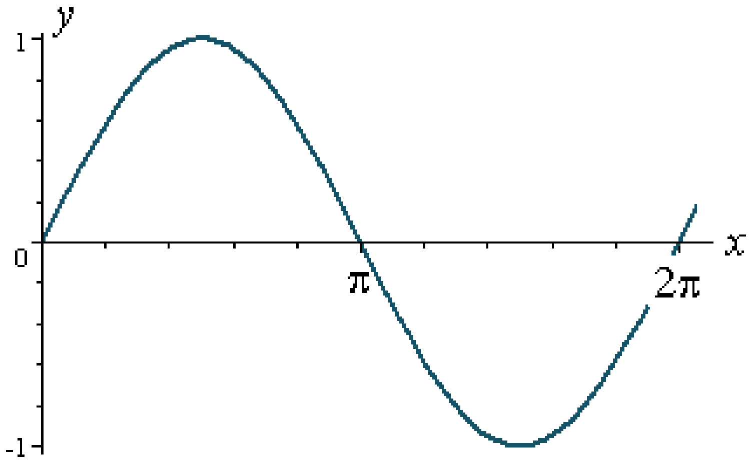
$y = a\sin bx$

Period = $2\pi/b$

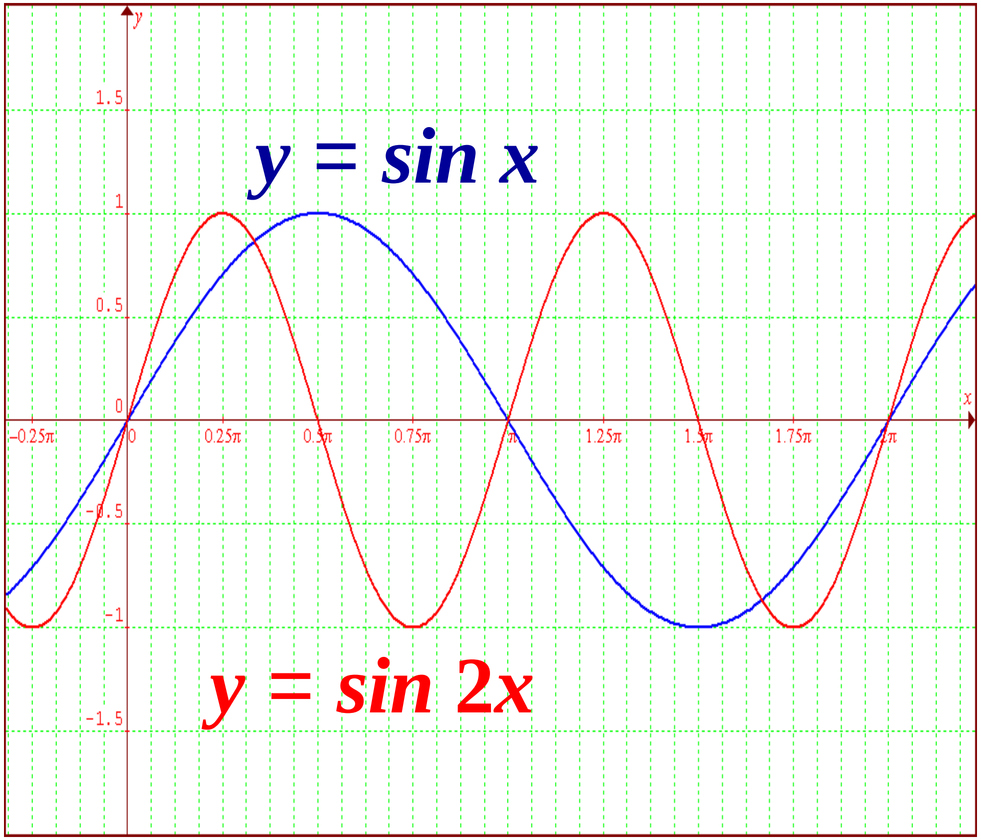
a = amplitude

We start with $y = \sin x$.

It has **amplitude** $= 1$ and **period** $= 2\pi$.

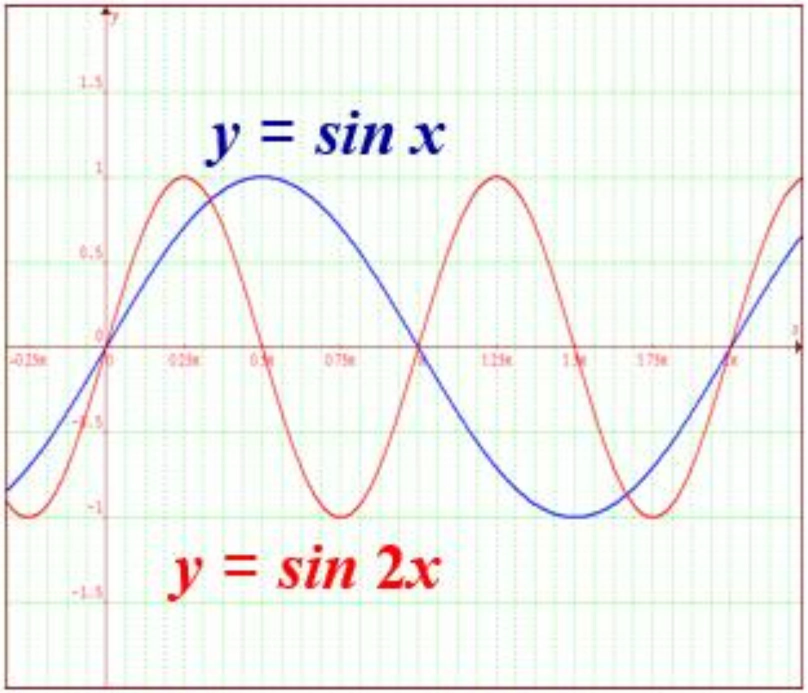


Period



- There are twice as many cycles.
- Once again, note the location of the 5 key points used in graphing.

Period



For y = sin ax

y = 0

when
$ax = 0, \pi, 2\pi$

or

$x = 0, \frac{\pi}{a}, \frac{2\pi}{a}$

For y = cos ax

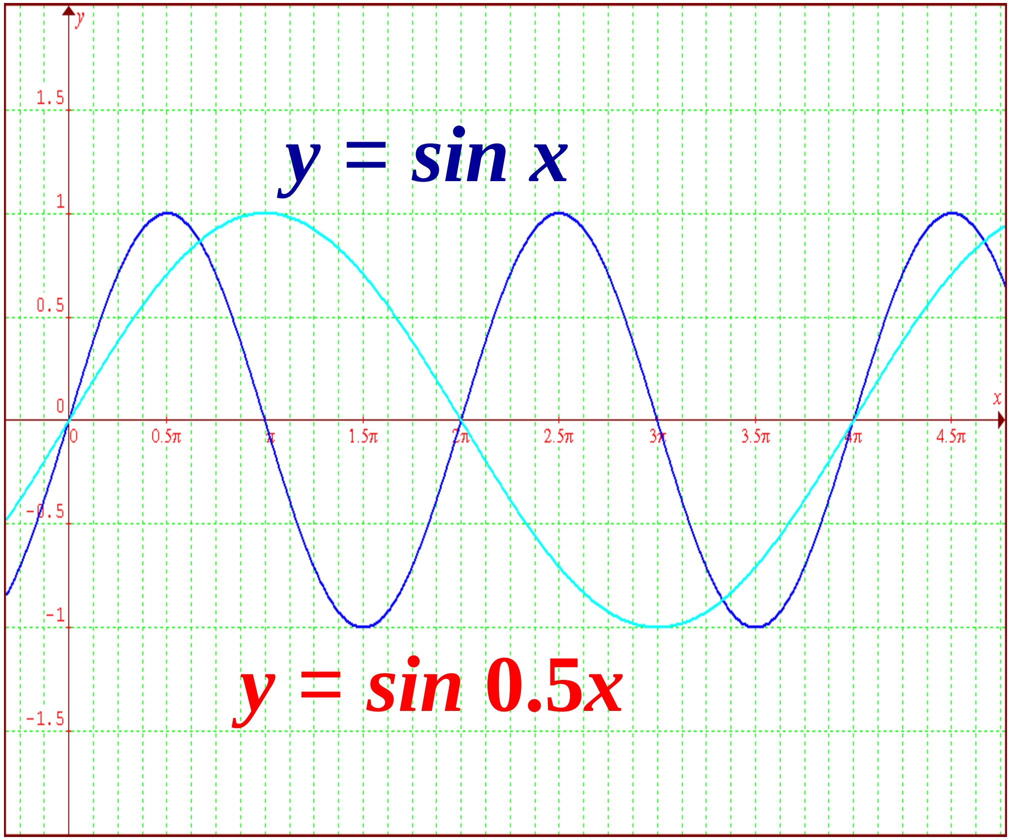
y = 0

when
$ax = \frac{\pi}{2}, \frac{3\pi}{2}$

or

$x = \frac{\pi}{2a}, \frac{3\pi}{2a}$

Period



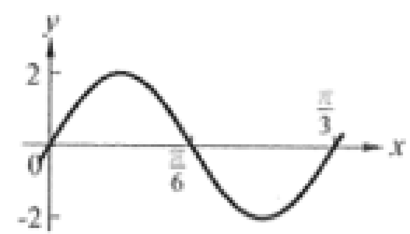
- There are half as many cycles here.

?

Ex 10.2 q 5 to 8, 23 to 25

23. $y = 2\sin 6x$ has amplitude of 2 and period $\frac{\pi}{3}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	0	2	0	-2	0



5. Since $\cos bx$ has period $\frac{2\pi}{b}$, $y = 3\cos 8x$ has a period of $\frac{2\pi}{8}$, or $\frac{\pi}{4}$.

A

5. Since $\cos bx$ has period $\frac{2\pi}{b}$, $y = 3\cos 8x$ has a period of $\frac{2\pi}{8}$, or $\frac{\pi}{4}$.

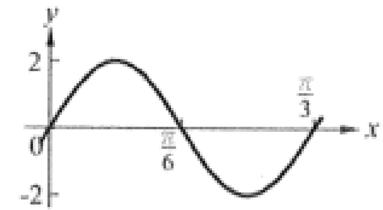
6. Since $\cos bx$ has period $\frac{2\pi}{b}$, $y = 28\cos 10x$ has a period of $\frac{2\pi}{10}$, or $\frac{\pi}{5}$.

7. $y = -2\sin 12x$ has period of $\frac{2\pi}{12} = \frac{\pi}{6}$.

8. $y = -\frac{1}{5}\sin 5x$ has period of $\frac{2\pi}{5}$.

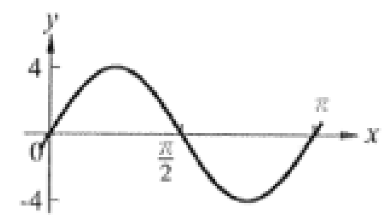
23. $y = 2\sin 6x$ has amplitude of 2 and period $\frac{\pi}{3}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	0	2	0	−2	0



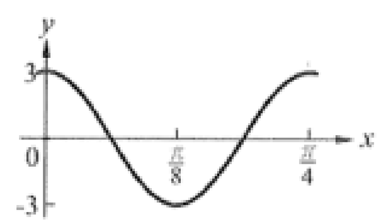
24. $y = 4\sin 2x$ has amplitude of 4 and period π.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2x$	0	1	0	−1	0
$4\sin 2x$	0	4	0	−4	0



25. $y = 3\cos 8x$ has amplitude of 3 and period $\frac{\pi}{4}$.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	3	0	−3	0	3

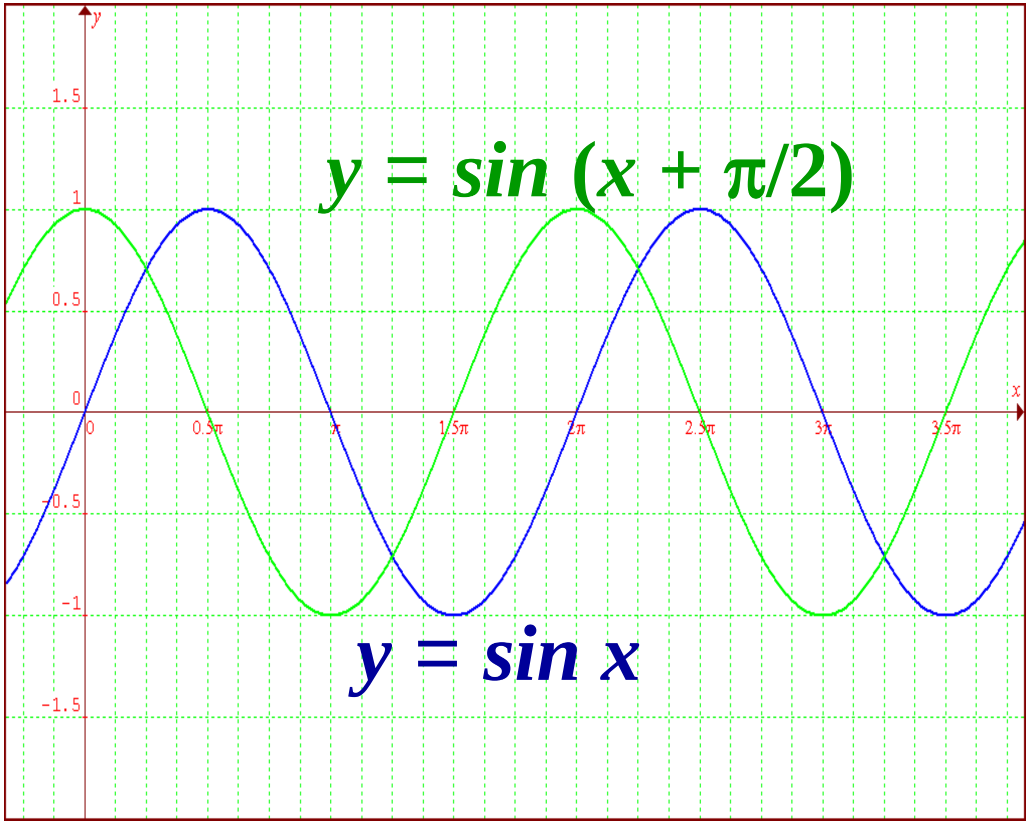


Ch. 10.3: Graphs of $y = a\sin(bx+c)$ & $y = a\cos(bx+c)$

- The effect of c in the equations $y = a\sin(bx+c)$ and $y = a\cos(bx+c)$
  - shift the curve to the ***left*** if $c > 0$, or
  - shift the curve to the ***right*** if $c < 0$.
- **Displacement or Phase Shift:**
  - The amount of shift is given by $-c/b$.

Displacement

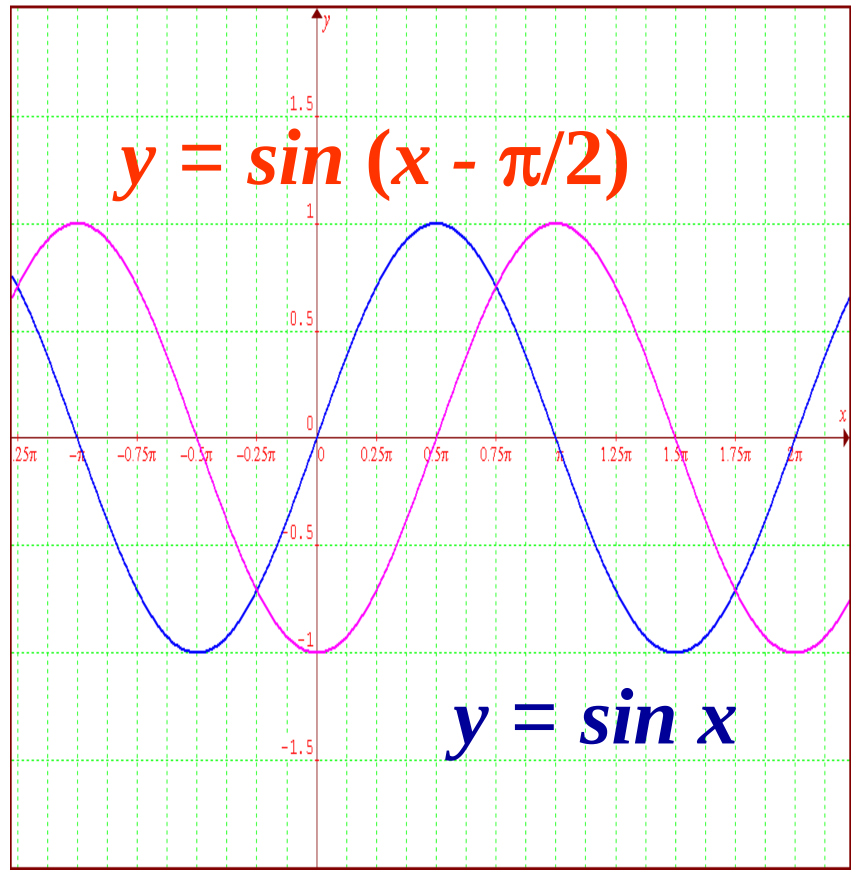
***y* = *a*sin(*b*x+*c*) & *y* = *a*cos(*b*x+*c*)**



- The displacement is ***negative*** (to the ***left***) for $c > 0$.

Displacement

- The displacement is ***positive*** (to the ***right***) for *c* < 0.



Summary

- Important Quantities to Determine for Sketching Graphs of $y = a\sin(bx + c)$ & $y = a\cos(bx + c)$:
  - Amplitude = $|a|$
  - Period = $2\pi/b$
  - Displacement = $-c/b$

For y=asin(bx+c)
amplitude= a, period = 2π/b, displacement = - c/b

EXAMPLE 2 Sketching graph of $y = a\sin(bx + c)$

Sketch the graph of $y = 2\sin(3x - \pi)$.

First, note that $a = 2$, $b = 3$, and $c = -\pi$. Therefore, the amplitude is 2, the period is $2\pi/3$, and the displacement is $-(-\pi/3) = \pi/3$. (We can also get the displacement from $3x - \pi = 0$, $x = \pi/3$.)

Note that the curve "starts" at $x = \pi/3$ and starts repeating $2\pi/3$ units to the right of this point. Be sure to grasp this point well. *The period tells us the number of units along the x-axis between such corresponding points.* One-fourth of the period is $\frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$.

Important values are at $\frac{\pi}{3}, \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}, \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$, and so on. We now make the table of important values and sketch the graph shown in Fig. 10.16.

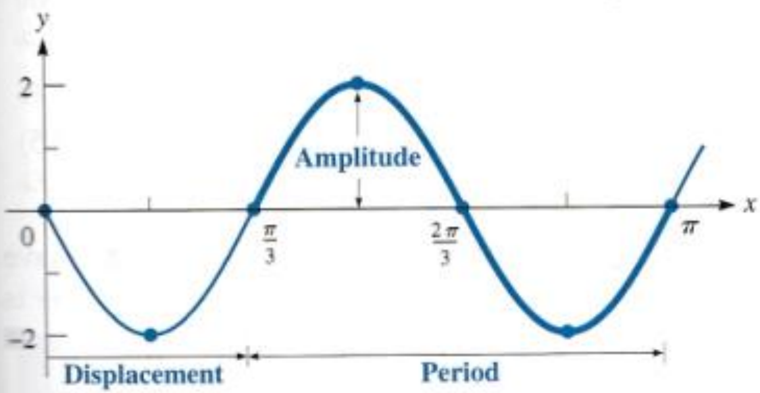


Fig. 10.16

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	−2	0	2	0	−2	0

$\leftarrow \frac{1}{4}(\text{period}) = \frac{\pi}{6}$

Note that since the period is $2\pi/3$, the curve passes through the origin.

?Ex 10.3 p298 q3-6, 23-26

In Exercises 3–26, determine the amplitude, period, and displacement for each function. Then sketch the graphs of the functions. Check each using a graphing calculator.

3. $y = \sin\left(x - \frac{\pi}{6}\right)$ **4.** $y = 3\sin\left(x + \frac{\pi}{4}\right)$

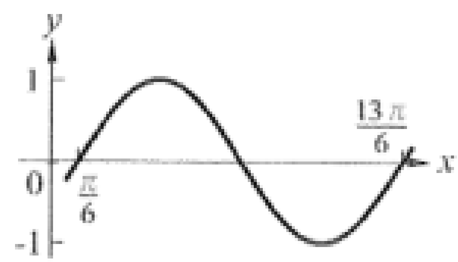
5. $y = \cos\left(x + \frac{\pi}{6}\right)$ **6.** $y = 2\cos\left(x - \frac{\pi}{8}\right)$

A

3. $y = \sin\left(x - \frac{\pi}{6}\right)$; $a = 1$, $b = 1$, $c = -\frac{\pi}{6}$

Amplitude is $|a| = 1$; period is $\frac{2\pi}{b} = 2\pi$;

displacement is $-\frac{c}{b} = \frac{\pi}{6}$

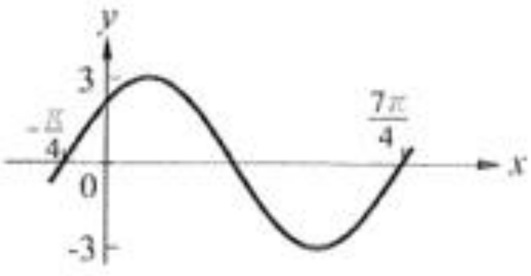


A

4. $y = 3\sin\left(x + \frac{\pi}{4}\right)$; $a = 3$, $b = 1$, $c = -\frac{\pi}{4}$

Amplitude is $|a| = 3$; period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$;

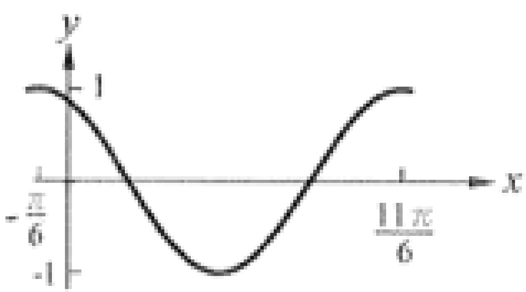
displacement is $-\frac{c}{b} = -\frac{\pi/4}{1} = -\frac{\pi}{4}$.



5. $y = \cos\left(x + \frac{\pi}{6}\right)$; $a = 1$, $b = 1$, $c = -\frac{\pi}{6}$

Amplitude is $|a| = 1$; period is $\frac{2\pi}{b} = 2\pi$;

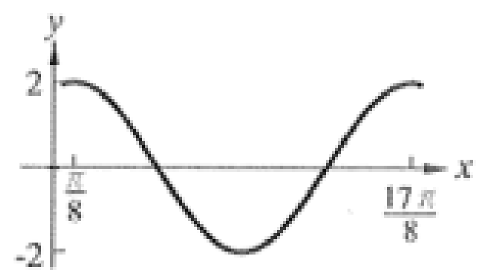
displacement is $-\frac{c}{b} = \frac{\pi}{6}$



6. $y = 2\cos\left(x - \frac{\pi}{8}\right)$; $a = 2$, $b = 1$, $c = \frac{-\pi}{8}$

Amplitude is $|a| = 2$; period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$;

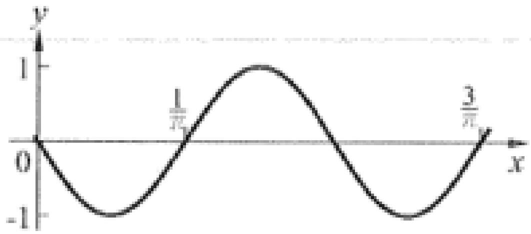
displacement is $-\frac{c}{b} = -\left(\frac{-\pi}{8}\right) = \frac{\pi}{8}$



23. $y=\sin\left(\pi^2x-\pi\right)$; $a=1$, $b=\pi^2$, $c=-\pi$

Amplitude is $|a|=1$; period is $\frac{2\pi}{b}=\frac{2}{\pi}$;

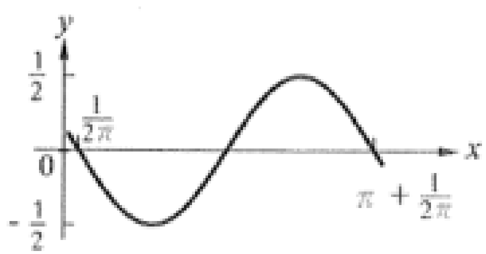
displacement is $-\frac{c}{b}=\frac{1}{\pi}$



24. $y=-\frac{1}{2}\sin\left(2x-\frac{1}{\pi}\right)$; $a=-\frac{1}{2}$, $b=2$, $c=-\frac{1}{\pi}$

Amplitude is $|a|=\frac{1}{2}$; period is $\frac{2\pi}{b}=\frac{2\pi}{2}=\pi$;

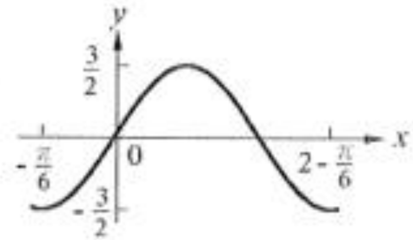
displacement is $-\frac{c}{b}=-\left(\frac{-1/\pi}{2}\right)=\frac{1}{2\pi}$



25. $y=-\frac{3}{2}\cos\left(\pi x-\frac{\pi^2}{6}\right)$; $a=-\frac{3}{2}$, $b=\pi$, $c=-\frac{\pi^2}{6}$

Amplitude is $|a|=\frac{3}{2}$; period is $\frac{2\pi}{b}=\frac{2\pi}{\pi}=2$;

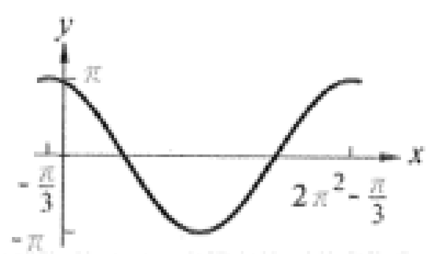
displacement is $-\frac{c}{b}=-\frac{\pi^2/6}{\pi}=-\frac{\pi}{6}$



26. $y=\pi\cos\left(\frac{1}{\pi}x+\frac{1}{3}\right)$; $a=\pi$, $b=\frac{1}{\pi}$, $c=\frac{1}{3}$

Amplitude is $|a|=\pi$; period is $\frac{2\pi}{b}=\frac{2\pi}{1/\pi}=2\pi^2$;

displacement is $-\frac{c}{b}=\frac{1/3}{1/\pi}=-\frac{\pi}{3}$

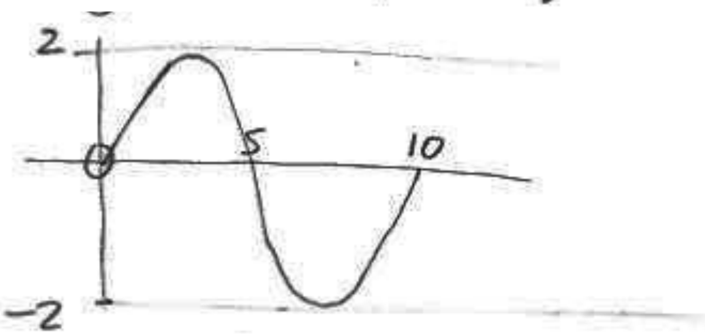


Graph → equation

$$y = a \sin(bx + c)$$

amplitude (a) Period $= \frac{2\pi}{b}$ Displacement $= -\frac{c}{b}$

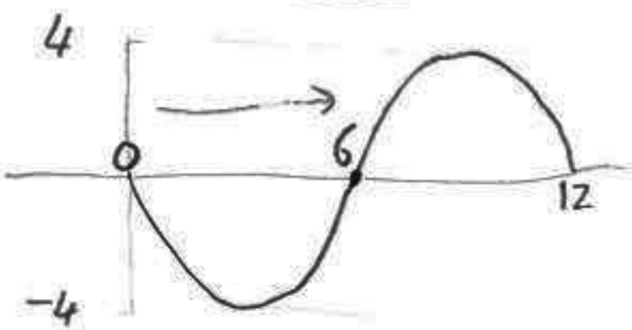
GRAPH TO EQUATIONS



$$y = 2\left(\sin\left(\frac{\pi}{5} + 0\right)\right)$$

$$10 = \frac{2\pi}{b} \Rightarrow b = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$y = -4\left(\sin\frac{\pi}{6}\right) = -2$$



$$y = 4\sin\left(\frac{\pi}{6} - \pi\right) \equiv -2$$

$$12 = \frac{2\pi}{b}$$

$$b = \frac{2\pi}{12} = \frac{\pi}{6}$$

Displacement

$$+6 = -\frac{c}{\alpha} = -\frac{c\,6}{\pi}$$

$$c = -\pi$$

Graph to equation Ex 10.3 q 41-44

41. $y = a\sin(bx+c)$. Amplitude $= 5$,

period $= \dfrac{2\pi}{b} = 16$, $b = \dfrac{\pi}{8}$

displacement $= -\dfrac{c}{b} = -1$, $c = \dfrac{\pi}{8}$

$y = 5\sin\left(\dfrac{\pi}{8}x + \dfrac{\pi}{8}\right)$

42. $y = a\cos(bx+c)$. Amplitude $= 5$,

period $= \dfrac{2\pi}{b} = 16$, $16b = 2\pi$; $b = \dfrac{\pi}{8}$

displacement $= -\dfrac{c}{b} = 3$, $c = \dfrac{3\pi}{8}$

$y = 5\cos\left(\dfrac{\pi}{8}x - \dfrac{3\pi}{8}\right)$

43. $y = a\cos(bx+c)$. Amplitude $= |-0.8| = 0.8$,

period $= \dfrac{2\pi}{b} = \pi$, $b = 2$

displacement $= -\dfrac{c}{b} = 0$, $c = 0$

$y = -0.8\cos 2x$

44. $y = a\sin(bx+c)$. Amplitude $= 0.8$,

period $= \dfrac{2\pi}{b} = \pi$, $b = 2$

displacement $= -\dfrac{c}{b} = \dfrac{\pi}{4}$, $c = -\dfrac{2\pi}{4} = -\dfrac{\pi}{2}$

$y = 0.8\sin\left(2x - \dfrac{\pi}{2}\right)$

Ch. 10.4: Graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \csc x$

- These functions are not defined for the values of x for which the curve has asymptotes.
- This means that the domains do not include the values of x at the points where the asymptotes exist.

Graph of *y* = *tan x*

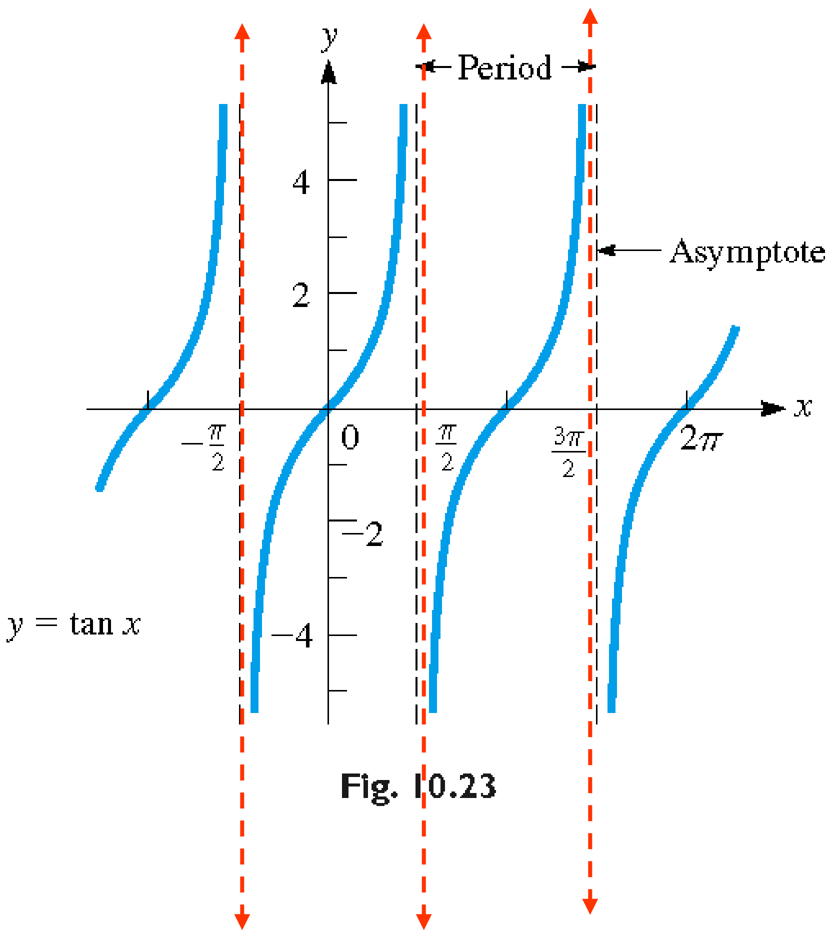


Fig. 10.23

- Note the asymptotes where ***y* = *tan x*** is undefined.
- Amplitude has no meaning here; period & phase shift do.
- Asymptote

is straight line that is closely approached by a plane curve so that the Perpendicular distance between them decreases to zero as the Distance from the Origin increases to infinity

Graph of *y* = *csc x*

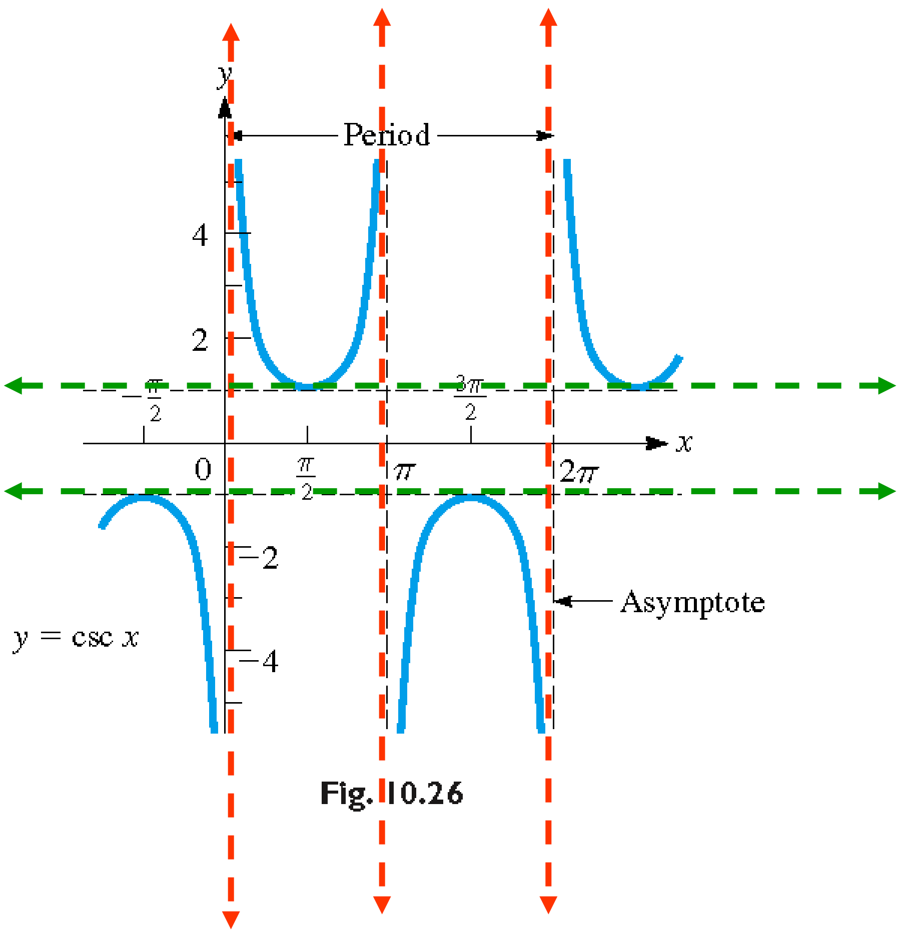


Fig. 10.26

- This graph is the reciprocal relationship of ***y* = *sin x***.
- Note the asymptotes where ***y* = *csc x*** is undefined.

Graph of *y* = *sec x*

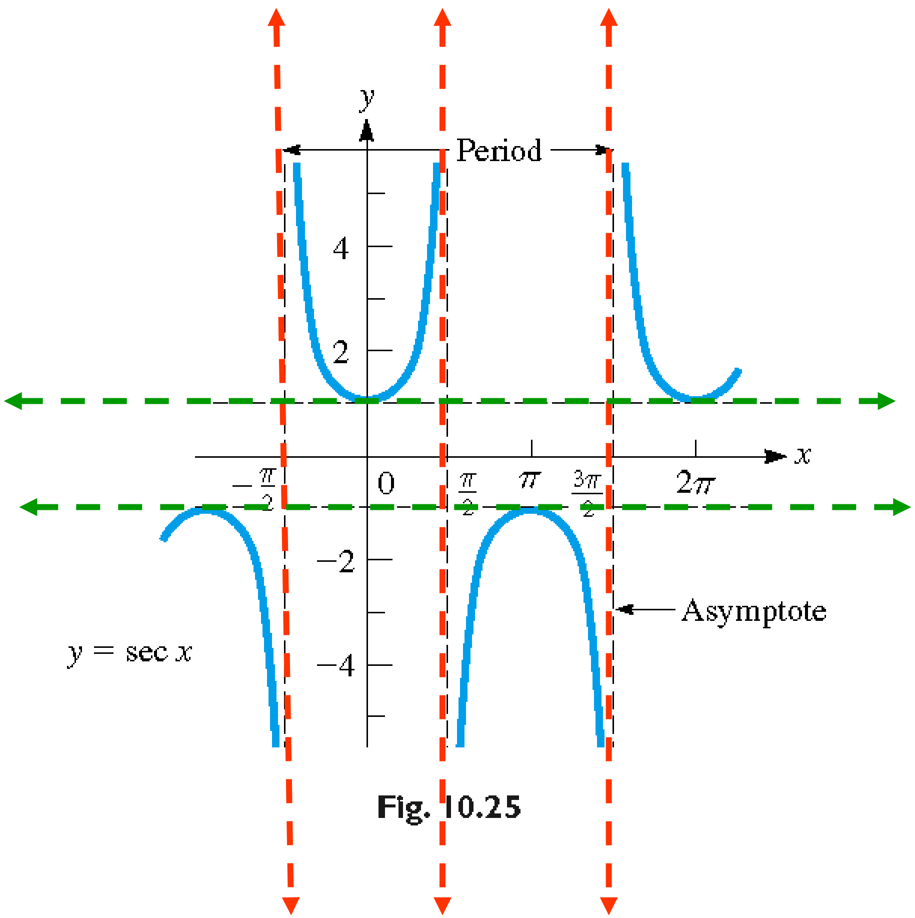


Fig. 10.25

- This graph is the reciprocal relationship of *y* = *cos x*.
- Note the asymptotes where *y* = *sec x* is undefined.

Graph of *y* = *cot x*

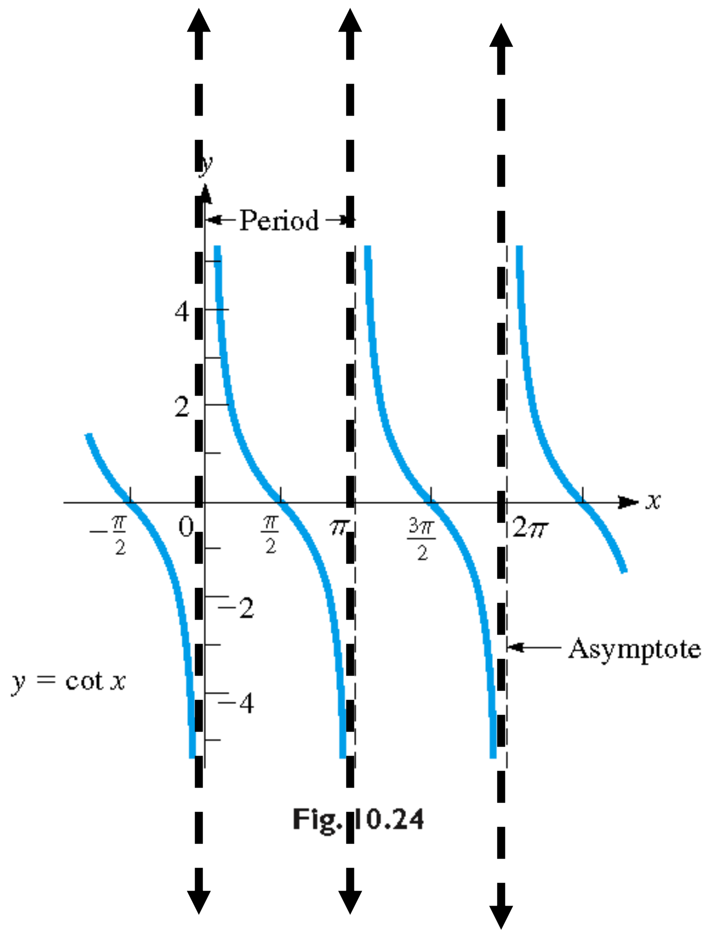


Fig. 10.24

- Note the asymptotes where ***y* = *tan x*** is undefined.
- Note the asymptotes are the same for ***y* = *cot x*** .

? Ex10.4 p301 q3-6

In Exercises 3–6, fill in the following table for each function and plot the graph from these points.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y													

3. $y = \tan x$

4. $y = \cot x$

5. $y = \sec x$

6. $y = \csc x$

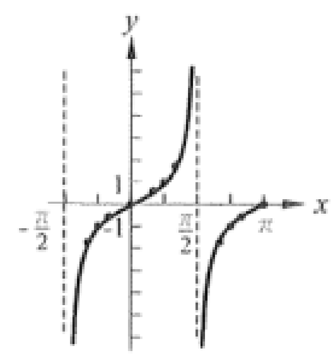
A

3.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
y	*	$-\sqrt{3}$	-1	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	$\sqrt{3}$	*	$-\sqrt{3}$	-1	$\frac{1}{-\sqrt{3}}$	0

$y = \tan x$



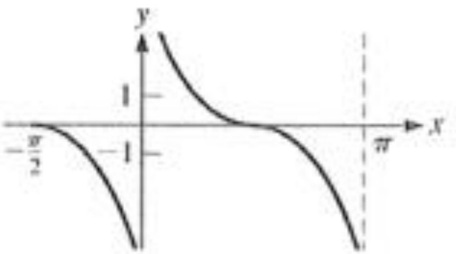
A

4.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$\cot x$	0	−0.58	−1	−1.7	*	1.7	1

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cot x$	0.58	0	−0.58	−1	−1.7	*

(* = undefined)

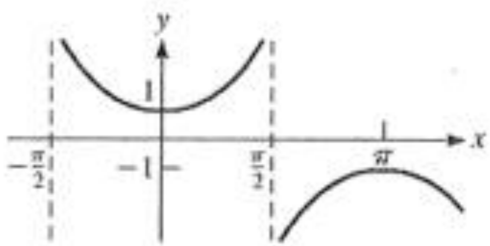


5.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$\sec x$	*	2	1.4	1.2	1	1.2	1.4

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sec x$	2	*	−2	−1.4	−1.2	−1

(* = undefined)

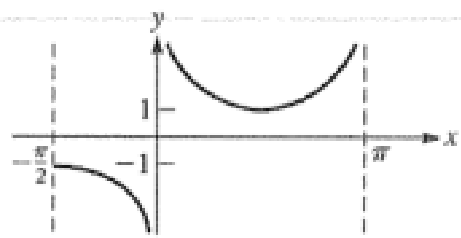


6.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$\cot x$	0	−0.58	−1	−1.7	*	1.7	1

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cot x$	0.58	0	−0.58	−1	−1.7	*

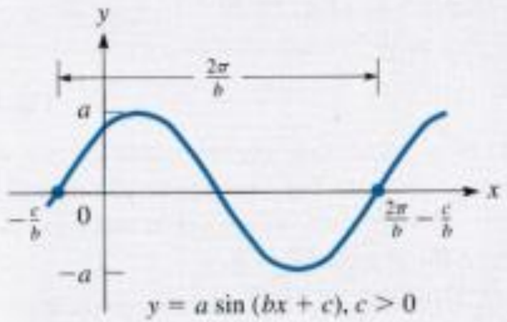
(* = undefined)



Summary

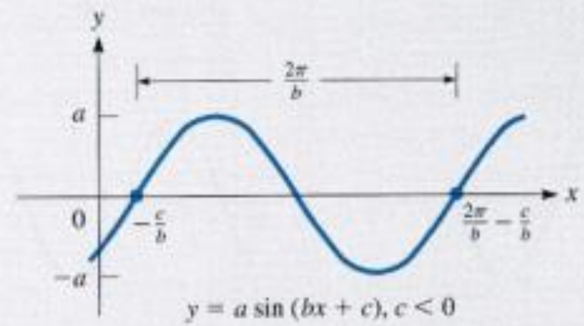
CHAPTER 10 EQUATIONS

For the graphs of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$ Amplitude $= |a|$ Period $= \dfrac{2\pi}{b}$ Displacement $= -\dfrac{c}{b}$ **(10.1)**



(a)

For each
$a > 0, b > 0$



(b)

Reciprocal relationships $\csc x = \dfrac{1}{\sin x}$ $\sec x = \dfrac{1}{\cos x}$ $\cot x = \dfrac{1}{\tan x}$ **(10.2)**

Chapter 22

Introduction to Statistics

Course Learning Outcomes

- 6) Perform basic statistical analysis including:
- 6.1 Compute the mean value
- 6.2 Compute the mode value
- 6.3 Compute the median value
- 6.4 Compute the standard deviation, percent error, confidence level, precision and accuracy, and distribution functions and INTERPRET the results.

When we deal with large amounts of data we can analyse them to find a lot of information.

This is statistics.

First we need to know some definitions and the ways we can represent data in graphs.

Ch. 22.1: Frequency Distributions

- Terminology:
  - ***Raw data***: numerical data collected but not yet organized
  - ***Array***: data that has been organized in either ascending or descending order
  - ***Classes***: groups used to organize the data

  - Eg people with red hair

Frequency Distributions

- ***Frequency***: the number of values in the class

Eg 4 people with red hair

- ***Frequency distribution table***: organizes the data according to class and frequency
- ***Relative frequency***: the frequency of the class is divided by the total frequency of all classes

See example 3 page 614

- ***Cumulative frequency***: obtained from adding the class frequencies

- ***Frequency***
- ***Frequency distribution table***:
- ***Relative frequency***:
- ***Cumulative frequency***:

Estimated Hours on Internet	*Frequency*	*Relative Frequency (%)*
0–4	2	4
5–9	9	18
10–14	19	38
15–19	11	22
20–24	6	12
25–29	3	6
Total	50	100

2/50 = 0.04 = 4%

Cumulative Freq
4
4+18
4+18+38
etc

Illustrating Frequencies

- ***Histogram***: a representation of a particular set of data by displaying each class of the data as a rectangle.
- ***Frequency polygon***: represents a set of data by plotting the x-values and the frequencies as the y-values

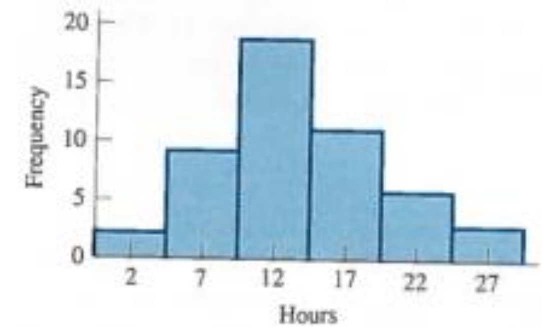


Fig. 22.1

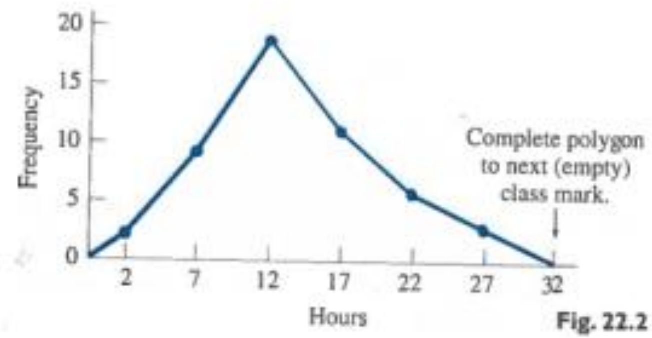


Fig. 22.2

Example

- The Talk-A-Lot Telephone Co. gathers data from 25 randomly selected customers. The length of time (in hours) each family spent on long distance calls on the July long weekend is gathered.
- The raw data are as follows:
  - 5.5, 4.1, 4.2, 6.5, 5.1, 2.4, 3.5, 4.7, 4.7, 3.5, 5.9, 6.6, 3.1, 5.1, 2.7, 4.4, 5.7, 4.4, 6.8, 5.9, 4.5, 5.5, 4.7, 4.9, 5.4

Example *(continued)*

- ***Construct***:
  1. A frequency distribution table
  2. A relative frequency distribution table
  3. A histogram
  4. A frequency polygon
  5. A cumulative frequency table
  6. An ogive of the data

Solution

5.5, 4.1, 4.2, 6.5, 5.1, 2.4, 3.5, 4.7, 4.7, 3.5,
5.9, 6.6, 3.1, 5.1, 2.7, 4.4, 5.7, 4.4, 6.8, 5.9,
4.5, 5.5, 4.7, 4.9, 5.4

1. The frequency distribution table:

Hours	**2.0-2.9**	**3.0-3.9**	**4.0-4.9**	**5.0-5.9**	**6.0-6.9**

5.5, 4.1, 4.2, 6.5, 5.1, 2.4, 3.5, 4.7, 4.7, 3.5, 5.9, 6.6, 3.1, 5.1, 2.7, 4.4, 5.7, 4.4, 6.8, 5.9, 4.5, 5.5, 4.7, 4.9, 5.4

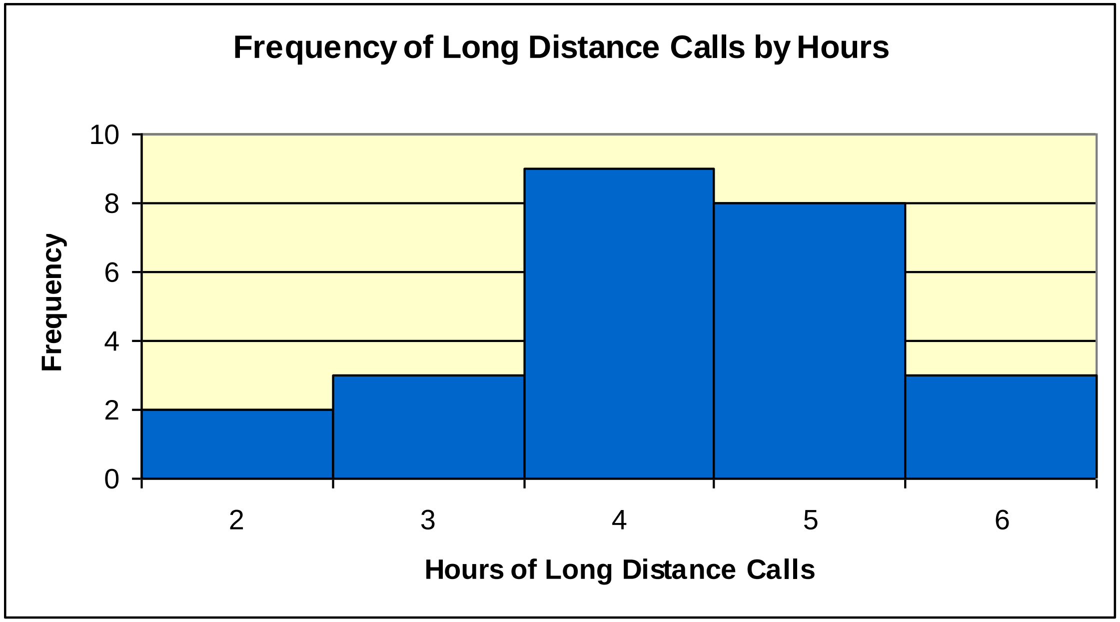
2. The relative frequency distribution table:

Hours	2.0-2.9	3.0-3.9	4.0-4.9	5.0-5.9	6.0-6.9
Frequency (families)	2	3	9	8	3

Solution

Hours	2.0-2.9	3.0-3.9	4.0-4.9	5.0-5.9	6.0-6.9
Frequency (families)	2	3	9	8	3

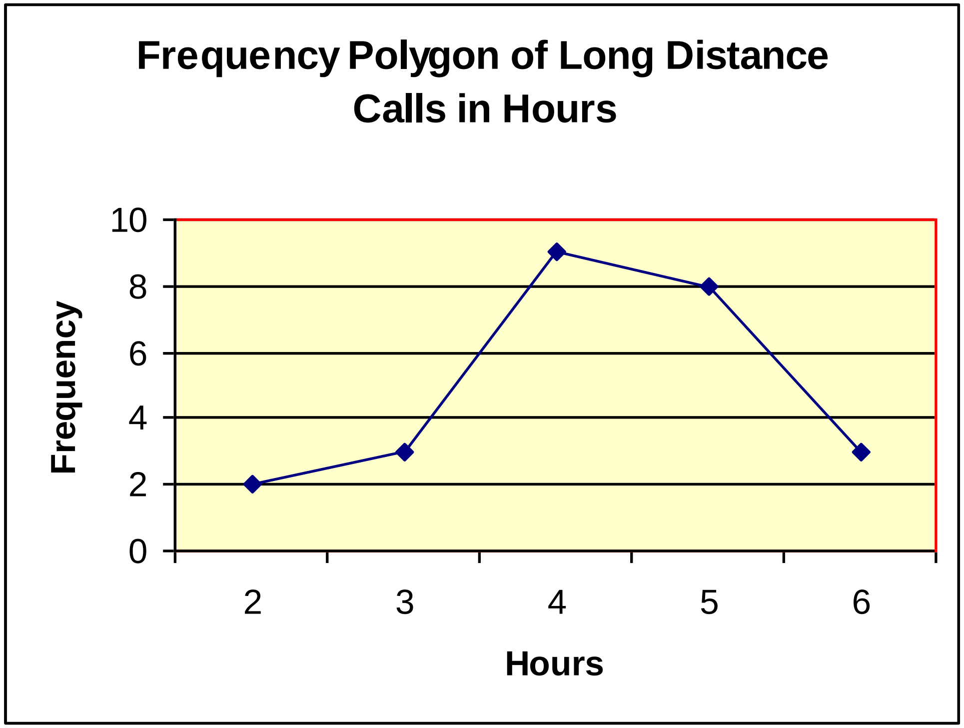
3. The histogram:



Solution

Hours	2.0-2.9	3.0-3.9	4.0-4.9	5.0-5.9	6.0-6.9
Frequency (families)	2	3	9	8	3

4. The frequency polygon:



Solution

Hours	2.0-2.9	3.0-3.9	4.0-4.9	5.0-5.9	6.0-6.9
Frequency (families)	2	3	9	8	3

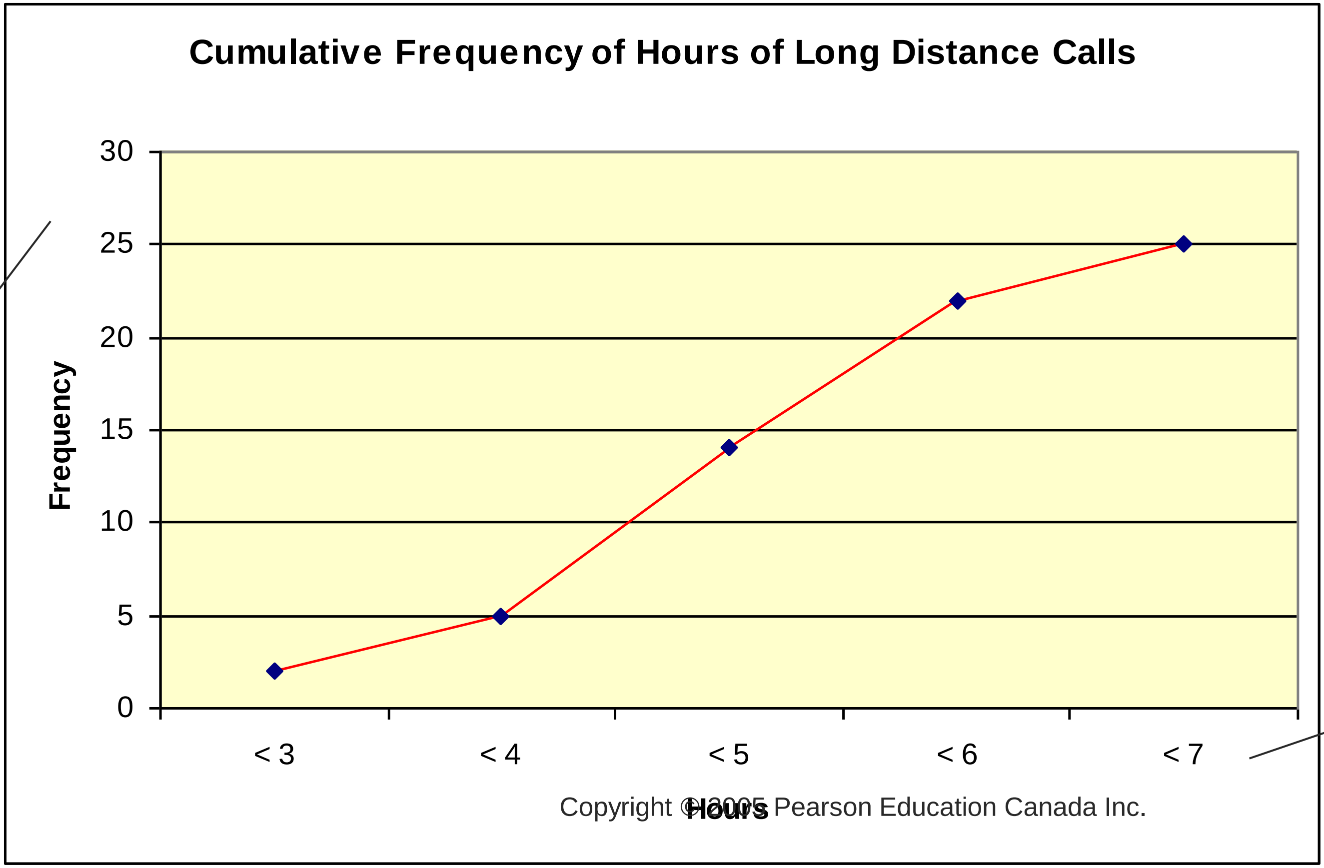
4. The cumulative frequency:

Hours on Long Distance Calls	Cumulative Frequency
< 3 hours	2
< 4 hours	5
< 5 hours	14
< 6 hours	22
< 7 hours	25

Solution

6. The ogive:

Hours on Long Distance Calls	Cumulative Frequency
< 3 hours	2
< 4 hours	5
< 5 hours	14
< 6 hours	22
< 7 hours	25



Frequency

Class boundaries

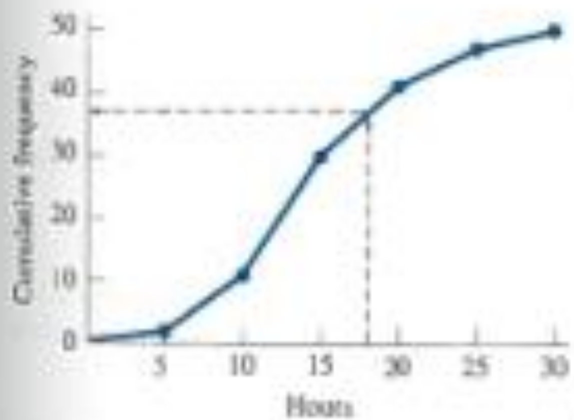


Fig. 22.4

Estimated Hours on Internet	*Cumulative Frequency*
Less than 5	2
Less than 10	11
Less than 15	30
Less than 20	41
Less than 25	47
Less than 30	50

The ogive showing the cumulative frequency for the values in this table is shown in Fig. 22.4. The vertical scale shows the *frequency*, and the horizontal scale shows the *class boundaries*.

? Ex22.1

In Exercises 5–12, use the following data. An automobile company tested a new engine, and found the following results in twenty tests of the number of miles traveled by a certain model car on each gallon of gasoline.

53, 58, 56, 54, 59, 54, 60, 58, 58, 54,
63, 56, 57, 56, 57, 59, 55, 61, 59, 58

5. Form a frequency distribution table.
6. Find the relative frequencies.
7. Form a frequency distribution table with five classes.
8. Find the relative frequencies for the data in the frequency distribution table in Exercise 7.
9. Draw a histogram for the data of Exercise 7.
10. Draw a frequency polygon for the data of Exercise 7.
11. Form a cumulative frequency table for the data of Exercise 7.
12. Draw an ogive for the data of Exercise 7.

A

53, 58, 56, 54, 59, 54, 60, 58, 58, 54,
63, 56, 57, 56, 57, 59, 55, 61, 59, 58

5.

Number	Frequency
53	1
54	3
55	1
56	3
57	2
58	4
59	3
60	1
61	1
62	0
63	1

6.

Number	Frequency	Relative frequency %
53	1	5
54	3	15
55	1	5
56	3	15
57	2	10
58	4	20
59	3	15
60	1	5
61	1	5
62	0	0
63	1	5
	10	100

1 x 100 / 20

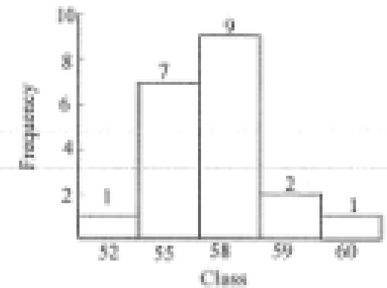
20

7.

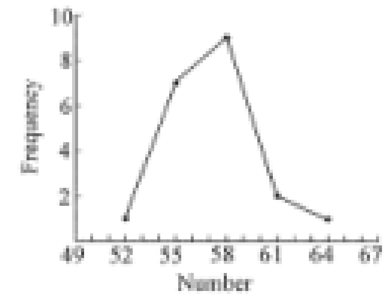
Class	Frequency	Relative frequency %
51 – 53	1	5
54 – 56	7	35
57 – 59	9	45
60 – 62	2	5
63 – 65	1	5
	20	100

8. See problem 7.

9.



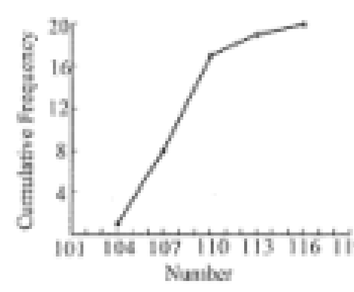
10.



11.

Number	Cummulative Freq.
54	1
57	8
60	17
63	19
66	20

12.



Now we know how to show data we want to find useful information from that data

Ch. 22.2: Measures of Central Tendency

- Common values used to measure the location of the center of the distribution:
  - Median
  - Mean
  - Mode

Median

- The ***median*** is the **middle number**, that number for which there are as many above it as below it in the distribution.
- If there is no middle number, the median is that number ***halfway*** between the two numbers nearest the middle of the distribution.

Example 1

- Seven employee salaries were recorded (in $1000s): **28, 60, 26, 32, 30, 26, 29**.
- What is the median salary?
- ***Steps***:
  1. Sort the salaries.
  2. Locate the value in the middle.

median

***Odd number of observations*: 26, 26, 28, 29, 30, 32, 60**

Example 2

- A sample of 10 adults was asked to report the number of hours they spent on the internet the previous month.
- The results in hours are as follows:

0 0 5 7 8 9 12 14 22 33

Solution 2

- ***Steps*:**
  1. Sort the hours.

Even number of observations

0 0 5 7 8 9 12 14 22 33

  1. Locate the value in the middle.

- We take the average of these values.
- ***Median*** **= (8 + 9)/2 = 8.5**

Arithmetic Mean

- This is the most popular and useful measure of central location

Mean = Sum of all the values / Number of values

$$\bar{x} = \frac{\Sigma xf}{\Sigma x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$

Example 3

- From Example 2, calculate the mean internet usage of the 10 adults.
- The number of hours are as follows:

0 7 12 5 33 14 8 0 9 22

- ***Solution:***

$$\bar{x} = \frac{0+7+12+5+33+14+8+0+9+22}{10} = 11.0$$

Mode

- The value that appears most frequently.
- There may be none, one or more than one.

Example 4

- From Example 2, calculate the mode of the internet usage of the 10 adults.

  0 7 12 5 33 14 8 0 9 22

- ***Answer*:**
- **Mode = 0.**
- When there are 2 modes, the data is said to be ***bimodal***.

median is the **middle number**

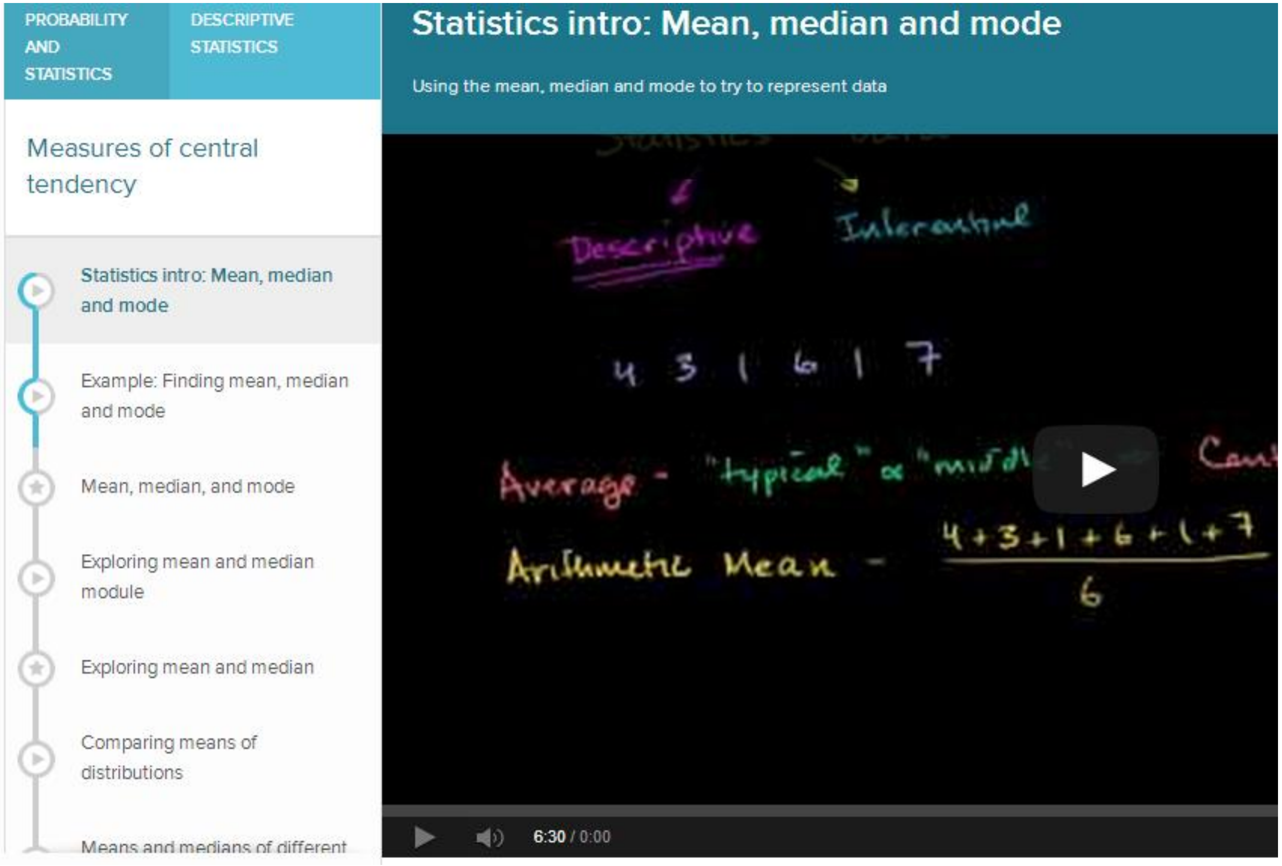
Mean = $\frac{\text{Sum of all the values}}{\text{Number of values}}$

$$\bar{x} = \frac{\Sigma xf}{\Sigma x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$

- Mode

The value that appears most frequently.

https://www.khanacademy.org/math/probability/descriptive-statistics/central_tendency/v/mean-median-and-mode



Plus many other examples

? From 5-16

EXERCISES 22.2

In Exercises 1–4, delete the 5 from the data numbers given for Example 1 and then do the following with the resulting data.

1. Find the median.
2. Find the arithmetic mean using the definition, as in Example 3.
3. Find the arithmetic mean using Eq. (22.1), as in Example 4.
4. Find the mode, as in Example 7.

In Exercises 5–16, use the following sets of numbers.

A: 3, 6, 4, 2, 5, 4, 7, 6, 3, 4, 6, 4, 5, 7, 3
B: 25, 26, 23, 24, 25, 28, 26, 27, 23, 28, 25
C: 0.48, 0.53, 0.49, 0.45, 0.55, 0.49, 0.47, 0.55, 0.48, 0.57, 0.51, 0.46, 0.53, 0.50, 0.49, 0.53
D: 105, 108, 103, 108, 106, 104, 109, 104, 110, 108, 108, 104, 113, 106, 107, 106, 107, 109, 105, 111, 109, 108

In Exercises 5–8, determine the median of the numbers of the given set.

5. Set *A*
6. Set *B*
7. Set *C*
8. Set *D*

In Exercises 9–12, determine the arithmetic mean of the numbers of the given set.

9. Set *A*
10. Set *B*
11. Set *C*
12. Set *D*

In Exercises 13–16, determine the mode of the numbers of the given set.

13. Set *A*
14. Set *B*
15. Set *C*
16. Set *D*

In Exercises 17–34, the required data are those in Exercises 22.1. Find the indicated measures of central tendency.

17. Median of miles per gallon of fuel usage in Exercise 5
18. Mean of miles per gallon of fuel usage in Exercise 5
19. Mode of miles per gallon of fuel usage in Exercise 5
20. Median of computer instructions in Exercise 13
21. Mean of computer instructions in Exercise 13
22. Mode of computer instructions in Exercise 13
23. Median of strobe light times in Exercise 17
24. Mean of strobe light times in Exercise 17
25. Median of stopping distances in Exercise 21. (Use the class mark for each class.)
26. Mean of stopping distances in Exercise 21. (Use the class mark for each class.)

A

5. Arrange the numbers in numerical order:
   2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 7, 7
   There are 15 numbers. The middle number is eighth. Since the eighth number is 4, the median is 4.

6. 23, 23, 24, 25, 25, 25, 26, 26, 27, 28, 28; $n = 11$;
   $M = 25$

7. Arrange in ascending order; $M = 0.495$;
   $n = 16\left(8^{th}, 9^{th}\right)$

8. Arrange in ascending order; $M = 107.5$;
   $n = 22\left(11^{th}, 12^{th}\right)$

9. The arithmetic mean is:
   $\bar{x}$
   $= \frac{2+3+3+3+4+4+4+4+5+5+6+6+6+7+7}{`5}$
   $= \frac{69}{15} = 4.6$

10. $\bar{x} = \frac{\sum xf}{\sum f} = \frac{280}{11} = 25.5$

11. $\bar{x} = \frac{\sum xf}{\sum f} = \frac{8.08}{16} = 0.505$

12. $\bar{x} = \frac{\sum xf}{\sum f} = \frac{2358}{22} = 107.2$

13. The mode is the number that occurs most frequently, which is 4 since it occurs 4 times.

14. $m = 25$ (occurs 3 times)

15. 0.49, 0.53 (occurs 3 times)

16. $m = 108$; $f = 5$

If we have a large amount of data we may want to find how far a certain value is from the mean

Ch. 22.3: Standard Deviation

- Terminology:
  - ***Population***: the complete collection of values
  - ***Sample***: a subset of the population

Standard Deviation

- The ***standard deviation*** indicates the spread of the data around the mean.
- The less spread in the data, the more reliable & descriptive it is.

$$s = \sqrt{\frac{\Sigma\left(x - \bar{x}\right)^2}{n-1}}$$

Steps for Calculating Standard Deviation

1. Find the arithmetic mean of the numbers of the set.
2. Subtract the mean from each number of the set.
3. Square these differences.
4. Find the sum of these differences.
5. Divide this sum by $n - 1$.
6. Find the square root of this result.

Example

- A sample of **10** adults was asked to report the number of hours they spent on the Internet the previous month.
- The results are as follows:

  0 0 5 7 8 9 12 14 22 33

- Calculate the standard deviation, ***s***, for this sample.
- Recall the sample mean was **11.0**.

0 0 5 7 8 9 12 14 22 33

sample mean was **11.0**.

- ***Steps***:

1. Find the arithmetic mean of the numbers of the set.
2. Subtract the mean from each number of the set.
3. Square these differences.

x	$x-\bar{x}$	$(x-\bar{x})^2$
0		
0		
5		
7		
8		
9		
12		
14		
22		
33		

Solution *(continued)*

4. Find the sum of these differences. $Sum = 917$

5. Divide this sum by $\boldsymbol{n-1}$. $\dfrac{\Sigma(x-\bar{x})^2}{n-1} = \dfrac{917}{9}$

6. Find the square root of this result. $s = \sqrt{\dfrac{917}{9}} = 10.1$

?[3-4

$$s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}}$$

$$s = \sqrt{\frac{n\left(\sum x^2\right) - \left(\sum x\right)^2}{n(n-1)}}$$

EXERCISES 22.3

In Exercises 1 and 2, in Example 1, change the first 1 to 6 and the first 2 to 7 and then find the standard deviation of the resulting data as directed.

1. Find s from the definition, as in Example 1.

2. Find s using Eq. 22.3, as in Example 3.

In Exercises 3–14, use the following sets of numbers. They are the same as those used in Exercise 22.2.

A: 3, 6, 4, 2, 5, 4, 7, 6, 3, 4, 6, 4, 5, 7, 3
B: 25, 26, 23, 24, 25, 28, 26, 27, 23, 28, 25
C: 0.48, 0.53, 0.49, 0.45, 0.55, 0.49, 0.47, 0.55, 0.48, 0.57, 0.51, 0.46, 0.53, 0.50, 0.49, 0.53
D: 105, 108, 103, 108, 106, 104, 109, 104, 110, 108, 108, 104, 113, 106, 107, 106, 107, 109, 105, 111, 109, 108

In Exercises 3–6, use Eq. (22.2) to find the standard deviation s for the indicated sets of numbers.

3. Set *A* **4.** Set *B* **5.** Set *C* **6.** Set *D*

In Exercises 7–10, use Eq. (22.3) to find the standard deviation s for the indicated sets of numbers.

7. Set *A* **8.** Set *B* **9.** Set *C* **10.** Set *D*

3.

x	$x-\bar{x}$	$(x-\bar{x})^2$
2	−260	6.76
3	−1.60	2.56
3	−1.60	2.56
3	−1.60	2.56
4	−0.60	0.36
4	−0.60	0.36
4	−0.60	0.36
4	−0.60	0.36
5	0.40	0.16
5	0.40	0.16
6	1.40	1.96
6	1.40	1.96
6	1.40	1.96
7	2.40	5.76
7	2.40	5.76
69		33.60

$\bar{x} = 69/15 = 4.60$

$33.60/14 = 2.4$

$s = \sqrt{2.4} = 1.55$

4.

x	$x-\bar{x}$	$(x-\bar{x})^2$	x^2
23	−2.455	6.0248	529
23	−2.455	6.0248	529
24	−1.455	2.1157	576
25	−0.4545	0.20661	625
25	−0.4545	0.20661	625
25	−0.4545	0.20661	625
26	0.54545	0.29752	676
26	0.54545	0.29752	676
27	1.5455	2.3884	729
28	2.5455	6.47973	784
28	2.5455	6.47973	784

$$\sum x = 280,\ \sum(x-\bar{x})^2 = 30.\overline{72},\ \sum x^2 = 7158$$

$$\bar{x} = \frac{\sum x}{n} = \frac{280}{11} = 25.\overline{45}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{30.\overline{72}}{11-1}} = 1.75,\ \text{equation 22.2}$$

The distribution of a large amount of data around the mean follows a pattern know as the normal distribution.

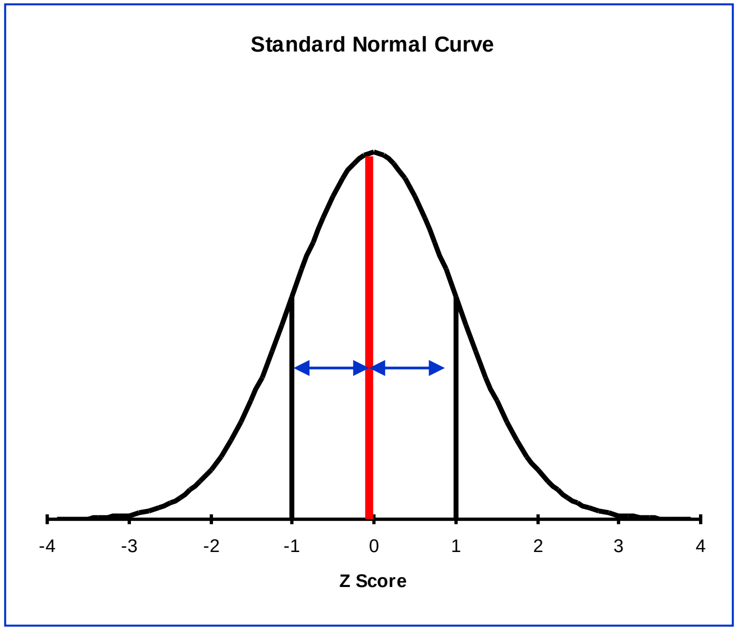
Ch. 22.4: Normal Distributions

- The ***normal distribution curve*** is a frequency polygon for a very large population with its maximum frequency very near the mean and tapering off to smaller frequencies on either side.

Standard Normal Distribution

- The ***standard normal distribution*** is the normal distribution for which the mean is **0** and the standard deviation, σ, is **1**.

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

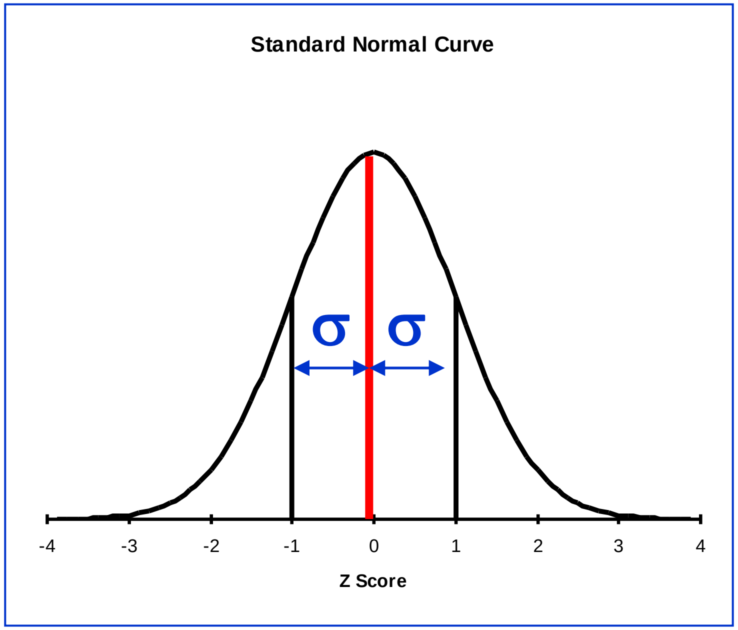


Characteristics of the Standard Normal Distribution

1. The curve is symmetric about the ***y***-axis.
2. Since the mean is **0**, the curve is symmetric about the mean.
3. The ***x***-axis is a horizontal asymptote.
4. The total area under the curve is **1**.

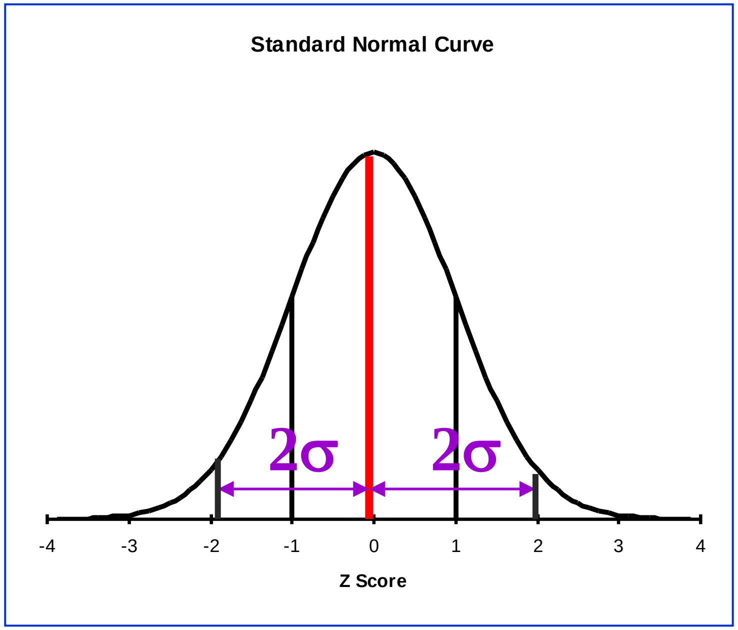
The Area Under the Curve

- Within one standard deviation, σ, each way of the mean
  - **68%** of the data



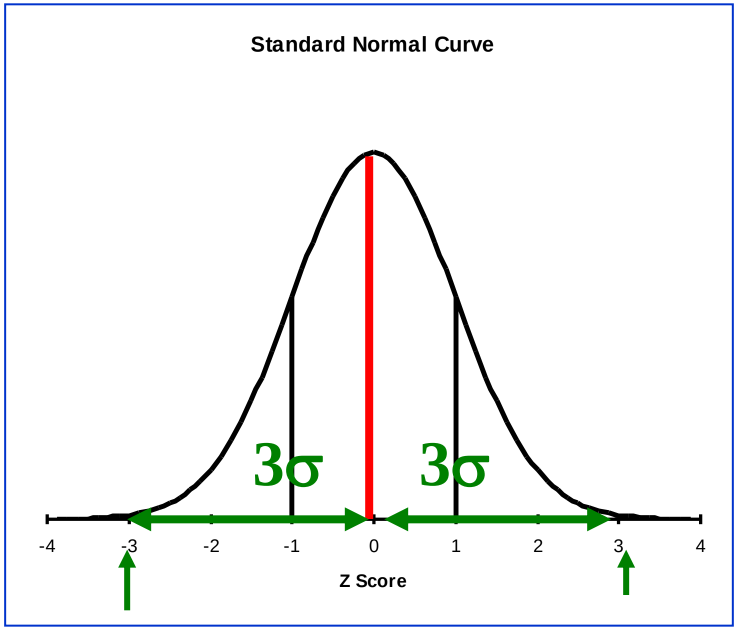
The Area Under the Curve

- Within two standard deviations, **2σ**, each way of the mean
  - **95%** of the data



The Area Under the Curve

- Within three standard deviations, **3σ**, each way of the mean
  - **99.7%** of the data



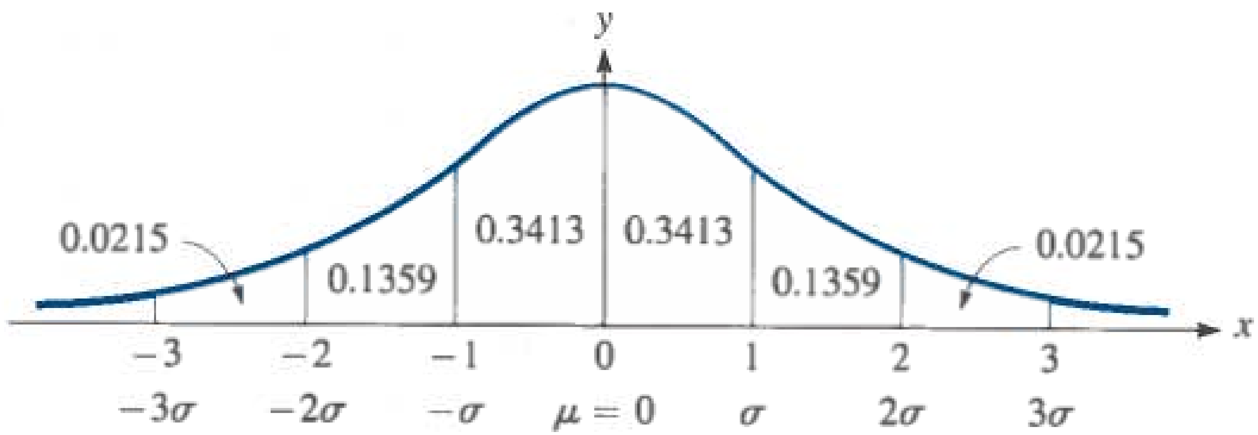


Fig. 22.11

From Fig. 22.11, we can see that *about 68% of the area is within one standard deviation of the mean (in the interval* $\mu - \sigma$ *to* $\mu + \sigma$*), and about 95% of the area is within two standard deviations of the mean (in the interval* $\mu - 2\sigma$ *to* $\mu + 2\sigma$*).* These percentages are often useful in data analysis.

Since a normal distribution curve gives a measure of the frequency of a particular value within the distribution, *the area under any part of the standard normal curve gives the relative frequency of those values of the distribution.* This is illustrated in the following example.

The Standard Normal Variable

- The standard normal curve with
  μ = mean = 0 and **σ = standard deviation = 1**
- x = the value on the x axis
- The number of ***standard deviations*** away from the ***mean*** is called:
  - the ***z*-score** or the ***Standard Score***.

$$z = \frac{x - \mu}{\sigma}$$

The *z*-score

- A value of ***z*** tells us the number of ***standard deviations*** the given value of ***x*** is above or below the ***mean***.

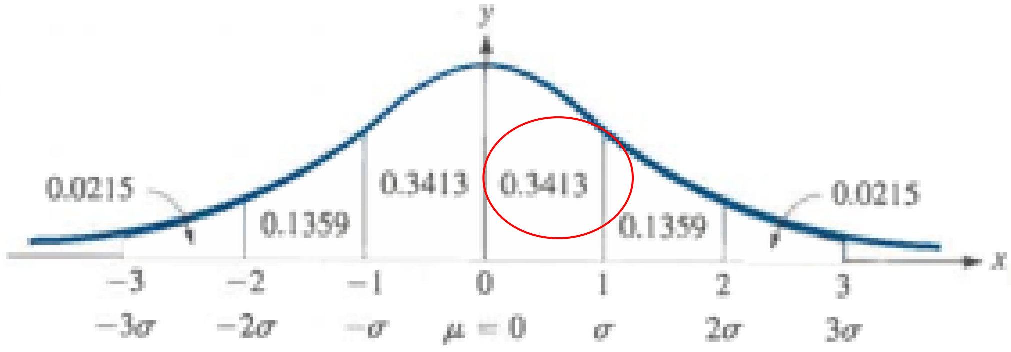


Fig. 22.11

Look up values on this table

On the next page, Table 22.1 gives the area under the standard normal distribution curve for the given values of z. The table includes values only to $z = 3$ since nearly all of the area is between $z = -3$ and $z = 3$. Since the curve is symmetric to the y-axis, the values shown are also valid for negative values of z.

Graph in table form

Real values to table

$$z = \frac{x - \mu}{\sigma}$$

Table 22.1 Standard Normal (z) Distribution

z	Area	z	Area	z	Area
0.0	0.0000	1.0	0.3413	2.0	0.4772
0.1	0.0398	1.1	0.3643	2.1	0.4821
0.2	0.0793	1.2	0.3849	2.2	0.4861
0.3	0.1179	1.3	0.4032	2.3	0.4893
0.4	0.1554	1.4	0.4192	2.4	0.4918
0.5	0.1915	1.5	0.4332	2.5	0.4938
0.6	0.2257	1.6	0.4452	2.6	0.4953
0.7	0.2580	1.7	0.4554	2.7	0.4965
0.8	0.2881	1.8	0.4641	2.8	0.4974
0.9	0.3159	1.9	0.4713	2.9	0.4981
1.0	0.3413	2.0	0.4772	3.0	0.4987

Example

- The mean water consumption for a community is **28.0** litres per week with standard deviation of **5.6** litres.
- What is the probability that you will find one family that has water usage between **28.0** litres and **33.04** litres?

0.9	0.3159

The mean water consumption for a community is **28.0** litres per week with standard deviation of **5.6** litres.

What is the probability that you will find one family that has water usage between **28.0** litres and **33.04** litres?

- The ***z*-score** represents the area under the normal distribution curve.
- This will give us the probability that we are looking for.

$$z = \frac{x - \mu}{\sigma} = \frac{33.04 - 28.0}{5.6} = 0.9$$

- For $\mathbf{z = 0.9}$, the probability that one family will use between **33.04** & **28.0** litres of water in one week is **0.3159** or **31.6%**.

Sometimes we need to know how far our data is from the normal curve.

Standard Error of the Mean

- **We assume the data is normally distributed.**
- To determine the variation of the data from a normally distributed data set, we calculate the standard error of the ***mean***.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

All possible samples of size ***n***.

Standard Error of *s*

- To determine the variation of the data from a normally distributed data set, we calculate the standard error of ***s***.

$$\sigma_s = \frac{\sigma}{\sqrt{2n}}$$

All possible samples of size ***n***.

- As the sample size gets larger ***(i.e. > 30)***, the less variation there should be in the mean & standard deviation of the sample.

? Ex 22.4 q9-12

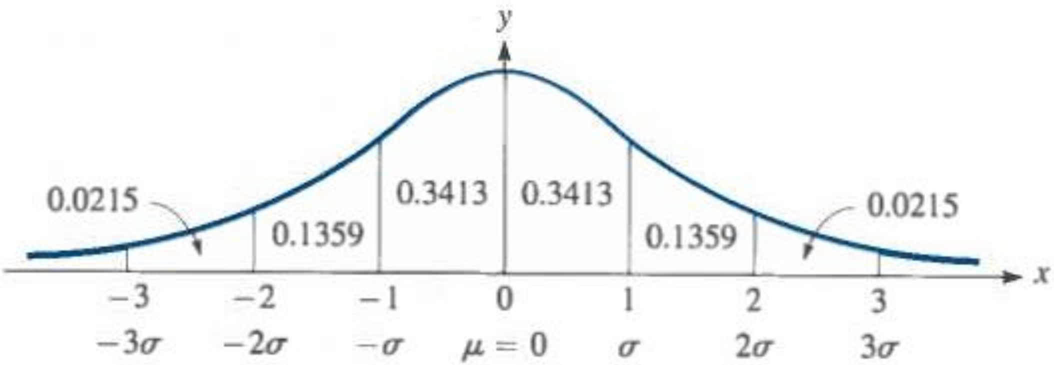
In Exercises 9–12, use the following data and refer to Fig. 22.11. A sample of 200 bags of cement are weighed as a quality check. Over a long period, it has been found that the mean value and standard deviation for this size bag are known and that the weights are normally distributed. Determine how many bags within this sample should have weights that satisfy the following conditions.

9. Within one standard deviation of the mean → 200 x 0.68 = 136

10. Within two standard deviations of the mean

11. Between the mean and two standard deviations above the mean

12. Between one standard deviation below the mean and three standard deviations above the mean



A

9. $200(0.68) = 136$ bags

10. $(2(0.1359) + 2(0.3413))(200) = 190.88$, about 190 bags

11. $(0.3413 + 0.1359)(200) = 95.44$, about 95 bags

12. $(0.1359 + 0.0215)(200) = 31.48$, about 31 bags X

Ch. 22.5: Statistical Process Control

- Statistical Process Control (SPC) is employed in industry to maintain and improve quality of products and services.
- No two products coming off an assembly line are exactly alike.
- Samples are tested during the production process at specified intervals to determine whether the production process needs adjustment to meet quality requirements.

Control

- A process is in ***control*** if it is stable and predictable.
  - Measurements fall within upper & lower control limits.
- A process is considered ***out of control*** if it has an unpredictable amount of variation.
  - Measurements fall outside the control limits due to special causes.

Control Charts

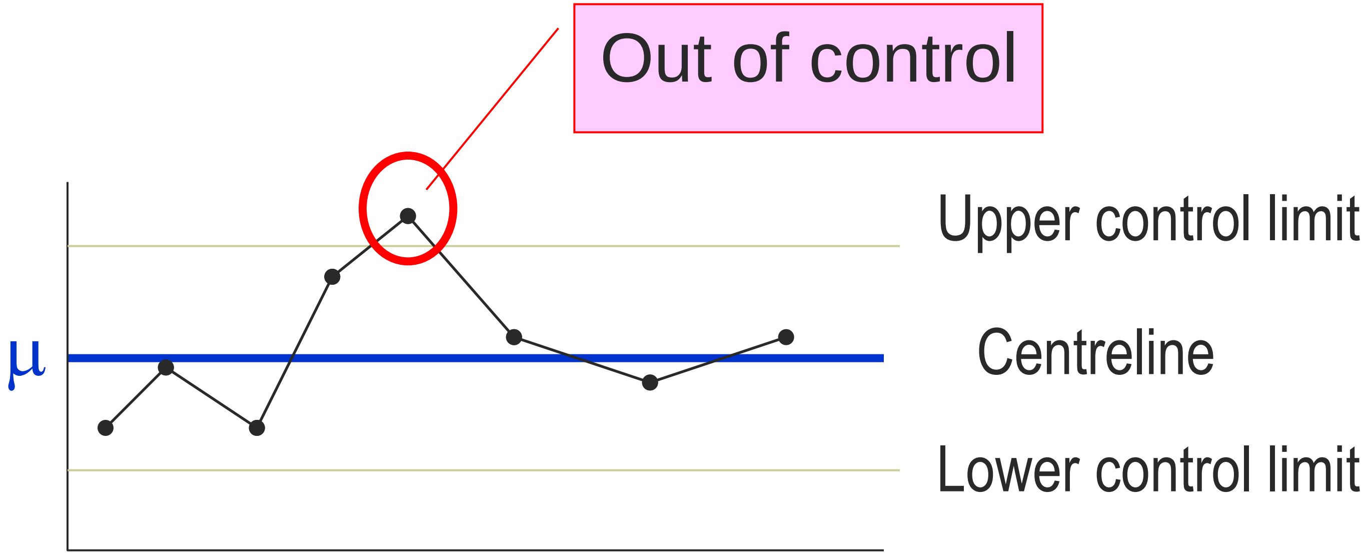
- Used to show a trend of a production characteristic over time.
- Samples are measured at specified intervals of time.
- Measurements are plotted on a chart to check for trends and abnormalities in the production process.

Key Features of Control Charts

- At a particular time in the production process, a sampling of measurements are taken.
- For each time period, the ***mean*** and ***range*** of the samples can be calculated.
- The ***range R*** of each sample is the difference between the highest value and the lowest value of the sample.

Key Features of Control Charts

- When we know the process distribution ***mean*** & variance.



Terminology used in SPC

- ***Variable***: a characteristic that can be measured
- ***Attribute***: a characteristic that can be counted
  - To monitor an attribute in a production process we calculate the proportion of defective parts.

EXAMPLE 2 Making $\bar{x}$ and R control charts

A pharmaceutical company makes a capsule of a prescrip
500 mg of the drug, according to the label. In a newly modi
the capsule, five capsules are tested every 15 min to check th
each capsule. Testing over a 5-h period gave the following resu
of samples.

Mean

Range

Sum

Means

Subgroup	Amount of Drug (in mg) of Five Capsules					Mean $\bar{x}$	Range R
1	503	501	498	507	502	502.2	9
2	497	499	500	495	502	498.6	7
3	496	500	507	503	502	501.6	11
4	512	503	488	500	497	500.0	24
5	504	505	500	508	502	503.8	8
6	495	495	501	497	497	497.0	6
7	503	500	507	499	498	501.4	9
8	494	498	497	501	496	497.2	7
9	502	504	505	500	502	502.6	5
10	500	502	500	496	497	499.0	6
11	502	498	510	503	497	502.0	13
12	497	498	496	502	500	498.6	6
13	504	500	495	498	501	499.6	9
14	500	499	498	501	494	498.4	7
15	498	496	502	501	505	500.4	9
16	500	503	504	499	505	502.2	6
17	487	496	499	498	494	494.8	12
18	498	497	497	502	497	498.2	5
19	503	501	500	498	504	501.2	6
20	496	494	503	502	501	499.2	9
					Sums	9998.0	174
					Means	499.9	8.7

As we noted from the table, *the **range** R of each sample is the difference between the highest value and the lowest value of the sample.*

From this table of values, we can make an $\bar{x}$ control chart and an R control chart. The $\bar{x}$ chart maintains a check on the average quality level, whereas the R chart maintains a check on the dispersion of the production process. These two control charts are often plotted together and referred to as the $\bar{x}$–R chart.

In order to define the **central line** of the $\bar{x}$ chart, which ideally is equivalent to the value of the population mean μ, we use the mean of the sample means $\bar{\bar{x}}$. For the central line of the R chart, we use $\bar{R}$. From the table, we see that

$$\bar{\bar{x}} = 499.9\text{ mg} \quad \text{and} \quad \bar{R} = 8.7\text{ mg}$$

The UCL and LCL for the $\bar{x}$ chart are found as follows:

$$\text{UCL}(\bar{x}) = \bar{\bar{x}} + A_2\bar{R} = 499.9 + 0.577(8.7) = 504.9\text{ mg}$$
$$\text{LCL}(\bar{x}) = \bar{\bar{x}} - A_2\bar{R} = 499.9 - 0.577(8.7) = 494.9\text{ mg}$$

The UCL and LCL for the R chart are found as follows:

$$\text{LCL}(R) = D_3\bar{R} = 0.000(8.7) = 0.0\text{ mg}$$
$$\text{UCL}(R) = D_4\bar{R} = 2.115(8.7) = 18.4\text{ mg}$$

Fig. 22.17 and the R control chart in Fig. 22.18.

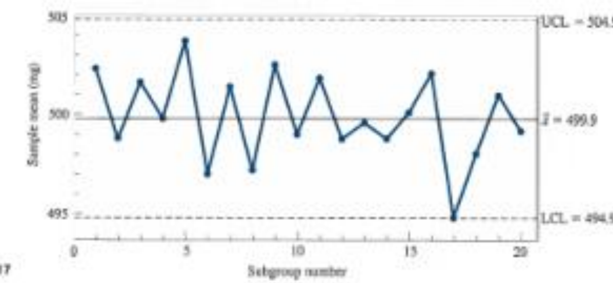


Fig. 22.17

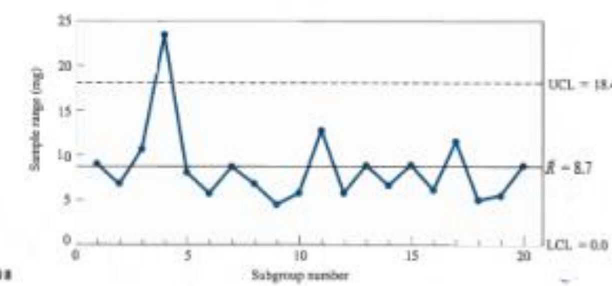


Fig. 22.18

This would be considered a *well-centered process* since $\bar{\bar{x}} = 499.9$ mg, which is very near the target value of 500.0 mg. We do note, however, that subgroup 17 was at a control limit and this might have been due to some special cause, such as the use of a substandard mixture of ingredients. We also note that the process was *out of control* due to some special cause of subgroup 4 since the range was above the upper control limit. We should keep in mind that there are numerous considerations, including human factors, that should be taken into account when making and interpreting control charts and that this is only a very brief introduction to this important industrial use of statistics. ■

Table 22.2 Control Chart Factors

n	d_2	A	A_2	D_1	D_2	D_3	D_4
5	2.326	1.342	0.577	0.000	4.918	0.000	2.115
6	2.534	1.225	0.483	0.000	5.078	0.000	2.004
7	2.704	1.134	0.419	0.205	5.203	0.076	1.924

5 samples so n=5

? Ex 22.5 5-8

Five automobile engines are taken from the production line each hour and tested for their torque (in N · m) when rotating at a constant frequency. The measurements of the sample torques for 20 h of testing are as follows:

Hour	*Torques (in N · m) of Five Engines*				
1	366	352	354	360	362
2	370	374	362	366	356
3	358	357	365	372	361
4	360	368	367	359	363
5	352	356	354	348	350
6	366	361	372	370	363
7	365	366	361	370	362
8	354	363	360	361	364
9	361	358	356	364	364
10	368	366	368	358	360
11	355	360	359	362	353
12	365	364	357	367	370
13	360	364	372	358	365
14	348	360	352	360	354
15	358	364	362	372	361
16	360	361	371	366	346
17	354	359	358	366	366
18	362	366	367	361	357
19	363	373	364	360	358
20	372	362	360	365	367

5. Find the central line, UCL, and LCL for the mean.

6. Find the central line, UCL, and LCL for the range.

7. Plot an $\overline{x}$ chart.

8. Plot an R chart.

5.

Hour	Torques	(N·m)	of five	engines	
1	366	352	354	360	362
2	370	374	362	366	356
3	358	357	365	372	361
4	360	368	367	359	363
5	352	356	354	348	350
6	366	361	372	370	363
7	365	366	361	370	362
8	354	363	360	361	364
9	361	358	356	364	364
10	368	366	368	358	360
11	355	360	359	362	353
12	365	364	357	367	370
13	360	364	372	358	365
14	348	360	352	360	354
15	358	364	362	372	361
16	360	361	371	366	346
17	354	359	358	366	366
18	362	366	367	361	357
19	363	373	364	360	358
20	372	362	360	365	367

A

Subgroup		Mean x	Range R
1		358.8	14
2		365.6	18
3		362.6	15
4		363.4	9
5		352.0	8
6		366.4	11
7		364.8	9
8		360.4	10
9		360.6	8
10		364.0	10
11		357.8	9
12		364.6	13
13		363.8	14
14		354.8	12
15		363.4	14
16		360.8	25
17		360.6	12
18		362.6	10
19		363.6	15
20		365.2	12
	Sum	7235.8	248
	Mean	361.79	12.4

$CL: \;\bar{x} = 361.79 \text{ N}\cdot\text{m}$

$UCL(\bar{x}) = \bar{x} + A_2\bar{R} = 361.79 + 0.577(12.4$

$= 368.9 \text{ N}\cdot\text{m}$

$LCL(\bar{x}) = \bar{x} - A_2\bar{R} = 361.79 - 0.577(12.4$

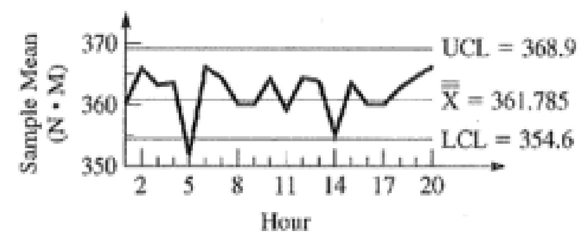
$= 354.6 \text{ N}\cdot\text{m}$

6. $CL: \;\bar{R} = 12.4 \text{ N}\cdot\text{m}$

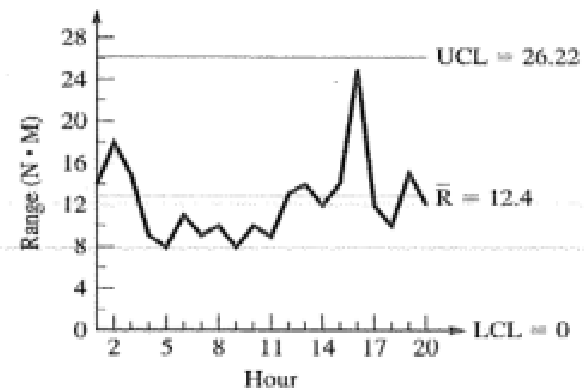
$LCL(\bar{R}) = D_3\bar{R} = 0.000(12.4) = 0.000 \text{ N}\cdot\text{m}$

$UCL(\bar{R}) = D_4\bar{R} = 2.115(12.4) = 26.2 \text{ N}\cdot\text{m}$

7.



8.



Ch. 22.6: Linear Regression

- We are finding the equation of a straight line that passes through a set of points.
- ***Regression***: fitting of a curve to a set of points.
- ***Linear regression***: fitting a set of points to a straight line.
- ***Nonlinear regression***: fitting a set of points to some other curve.

Linear Regression

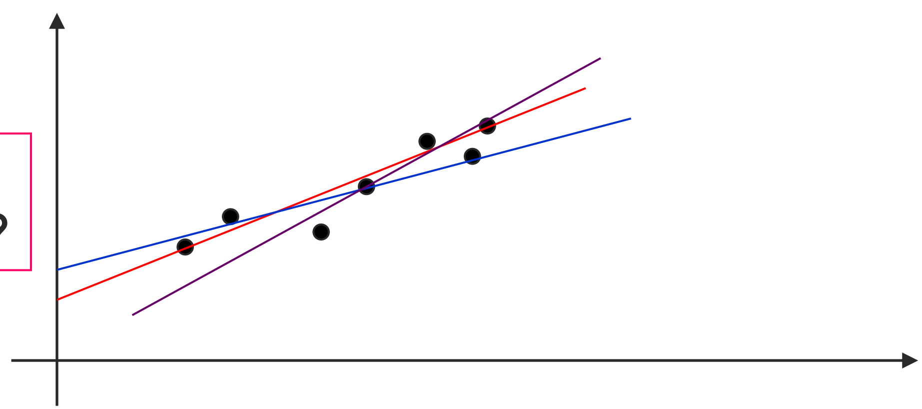
Why linear regression?

1. to express a concise relationship between variables,
2. to use the equation to predict certain fundamental results,
3. to determine the reliability of certain sets of data,
4. to use the data for testing certain theoretical concepts.

Linear Regression

- We are working with a set of at least **5** or **6** points.
- We cannot reasonably expect that a line will pass through all points exactly.

The question is:
Which straight line fits best?



Method of Least Squares

- The sum of the squares of the ***deviations*** of all data points from the best line (in accordance with this method) has the least value possible.
- ***Deviation***: the difference between the y-value of the line and the y-value for the point (of original data) for a particular value of x.

Method of Least Squares

- The equation of the least-squares line:

$$y = mx + b$$

$$m = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma x^2 - (\Sigma x)^2}$$

$$b = \frac{(\Sigma x^2)(\Sigma y) - (\Sigma xy)(\Sigma x)}{n\Sigma x^2 - (\Sigma x)^2}$$

- The equation of the least-squares line:

$$y = mx + b$$

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2}$$

$$b = \frac{(\sum x^2)(\sum y) - (\sum xy)(\sum x)}{n\sum x^2 - (\sum x)^2}$$

EXAMPLE 3 Finding equation of least-squares line

Find the equation of the least-squares line for the points indicated in the following table. Graph the line and data points on the same graph.

x	1	2	3	4	5
y	3	6	6	8	12

We see from Eqs. (22.10) and (22.11) that we need the sums of x, y, xy, and x^2 in order to find m and b. Thus, we set up a table for these values, along with the necessary calculations, as follows:

	x	y	xy	x^2
	1	3	3	1
	2	6	12	4
	3	6	18	9
	4	8	32	16
	5	12	60	25
sums →	15	35	125	55
	↑ $\sum x$	↑ $\sum y$	↑ $\sum xy$	↑ $\sum x^2$

$n = 5 \quad (5 \text{ points})$

$$m = \frac{5(125) - (15)(35)}{5(55) - (15)^2} = \frac{100}{50} = 2$$

$$b = \frac{(55)(35) - (125)(15)}{50} = \frac{50}{50} = 1$$

This means that the equation of the least-squares line is $y = 2x + 1$. This line and the data points are shown in Fig. 22.23.

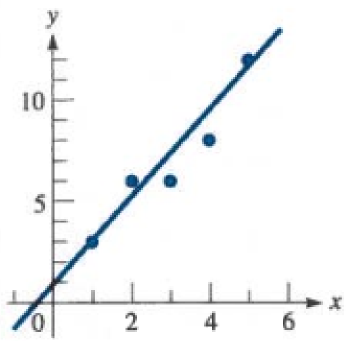


Fig. 22.23

? Ex 22.6 q7

EXERCISES 22.6

In Exercises 1–14, find the equation of the least-squares line for the given data. Graph the line and data points on the same graph.

1. In Example 3, replace the y-values with 3, 7, 9, 9, and 12. Then follow the instructions above.

2.

x	1	2	3	4	5	6	7
y	10	17	28	37	49	56	72

3.

x	20	26	30	38	48	60
y	160	145	135	120	100	90

4.

x	1	3	6	5	8	10	4	7	3	8
y	15	12	10	8	9	2	11	9	11	7

5. In Example 5, change the y (mg of drug/dL of blood) values to 8.7, 8.4, 7.7, 7.3, 5.7, 5.2. Then proceed to find y as a function of t, as in Example 5.

6. The velocity v (in m/s) of a falling object was found each second by use of an electronic device, as shown in the following table. Find v as a function of t.

t (s)	1.00	2.00	3.00	4.00	5.00	6.00	7.00
v (m/s)	9.70	19.5	29.5	39.4	49.2	58.9	68.6

7. In an electrical experiment, the following data were found for the values of current and voltage for a particular element of the circuit. Find the voltage V as a function of the current i.

Current (mA)	15.0	10.8	9.30	3.55	4.60
Voltage (V)	3.00	4.10	5.60	8.00	10.50

8. A particular muscle was tested for its speed of shortening as a function of the force applied to it. The results appear below. Find the speed as a function of the force.

Force (N)	60.0	44.2	37.3	24.2	19.5
Speed (m/s)	1.25	1.67	1.96	2.56	3.05

A

7.

i	V	iV	i^2
15.0	3.00	45.00	225.00
10.8	4.10	44.28	116.64
9.30	5.60	52.08	86.49
3.55	8.00	28.40	12.60
4.60	10.50	48.30	21.16
43.25	31.20	218.06	461.89

$n = 5$

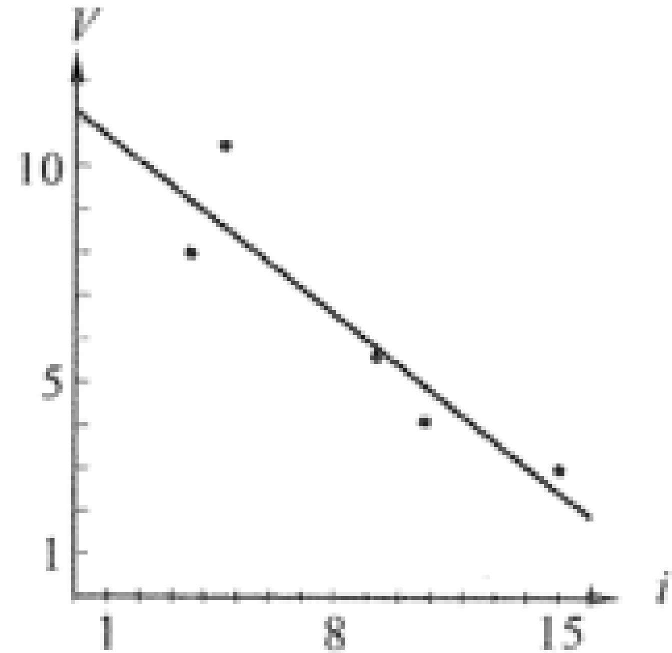
$$m = \frac{5(218.06) - (43.25)(31.20)}{5(461.89) - (43.25)^2} = -0.590$$

$$b = \frac{(461.89)(31.20) - (218.06)(43.25)}{5(461.89) - (43.25)^2} = 11.3$$

$V = mi + b; V = -0.590i + 11.3$

Plot points:

i	V
2	10.1
10	5.3



Ch. 22.7: Nonlinear Regression

- ***Nonlinear regression***: fitting a set of points to some other curve.
- From a plot of points, we may recognize a different type of curve.
- We extend the method of least squares to these other curves.

Extending Linear Regression to Nonlinear Regression

The least squares line in linear regression

$$y = mx + b$$

The least squares line in nonlinear regression

$$y = m\left[f\left(x\right)\right] + b$$

$$y = m\left[f\left(x\right)\right] + b$$

$$m = \frac{n\Sigma f(x)y - \left(\Sigma f(x)\right)\left(\Sigma y\right)}{n\Sigma(f(x))^2 - \left(\Sigma f(x)\right)^2}$$

$$b = \frac{\left(\Sigma f(x)^2\right)\left(\Sigma y\right) - \left(\Sigma f(x)y\right)\left(\Sigma f(x)\right)}{n\Sigma(f(x))^2 - \left(\Sigma f(x)\right)^2}$$

Using Nonlinear Regression Equations

- We calculate the function first then deal with the problem as a least-squares line to find the values of m and b.
- Some functions are:
  - Quadratic: x^2
  - Hyperbolic: x^{-1} (or $1/x$)
  - Exponential: 10^x

$$y = m[f(x)] + b$$

$$m = \frac{n\Sigma f(x)y - (\Sigma f(x))(\Sigma y)}{n\Sigma(f(x))^2 - (\Sigma f(x))^2}$$

$$b = \frac{(\Sigma f(x)^2)(\Sigma y) - (\Sigma f(x)y)(\Sigma f(x))}{n\Sigma(f(x))^2 - (\Sigma f(x))^2}$$

EXAMPLE 1 Fitting $y = mx^2 + b$ to set of points

Find the least-squares curve $y = mx^2 + b$ for the following points:

x	0	1	2	3	4	5
y	1	5	12	24	53	76

In using Eq. (22.12), $f(x) = x^2$. Our first step is to calculate values of x^2, and then we use x^2 as we used x in finding the equation of the least-squares line.

x	$f(x) = x^2$	y	x^2y	$(x^2)^2$
0	0	1	0	0
1	1	5	5	1
2	4	12	48	16
3	9	24	216	81
4	16	53	848	256
5	25	76	1900	625
	55	171	3017	979

$$n = 6$$

$$m = \frac{6(3017) - 55(171)}{6(979) - 55^2} = 3.05$$

$$b = \frac{(979)(171) - (3017)(55)}{6(979) - 55^2} = 0.52$$

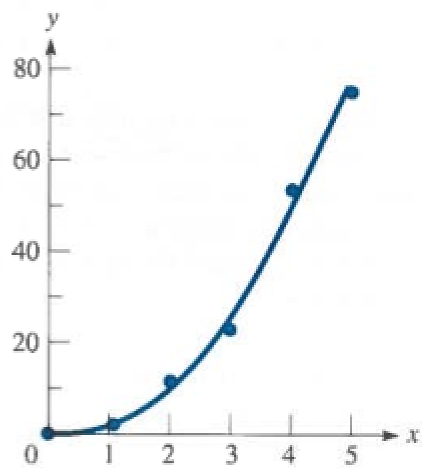


Fig. 22.27

Therefore, the required equation is $y = 3.05x^2 + 0.52$. The graph of this equation and the data points are shown in Fig. 22.27.

? Ex 22.7 q 6

2. For the points in the following table, find the least-squares curve $y = m\sqrt{x} + b$.

x	0	4	8	12	16
y	1	9	11	14	15

$$y = m[f(x)] + b$$

$$m = \frac{n\Sigma f(x)y - (\Sigma f(x))(\Sigma y)}{n\Sigma(f(x))^2 - (\Sigma f(x))^2}$$

$$b = \frac{(\Sigma f(x)^2)(\Sigma y) - (\Sigma f(x)y)(\Sigma f(x))}{n\Sigma(f(x))^2 - (\Sigma f(x))^2}$$

6. $y = m(x^2) + b$

x	y	x^2	yx^2	$(x^2)^2$
50.0	1.00	2500	2500	6,250,000
100	4.40	10,000	44,000	1×10^8
150	9.40	22,500	211,500	5.0625×10^8
200	16.4	40,000	656000	16×10^8
250	24.0	62,500	1,500,000	39.0625×10^8
	55.2	137,500	2,414000	61.1875×10^8

A

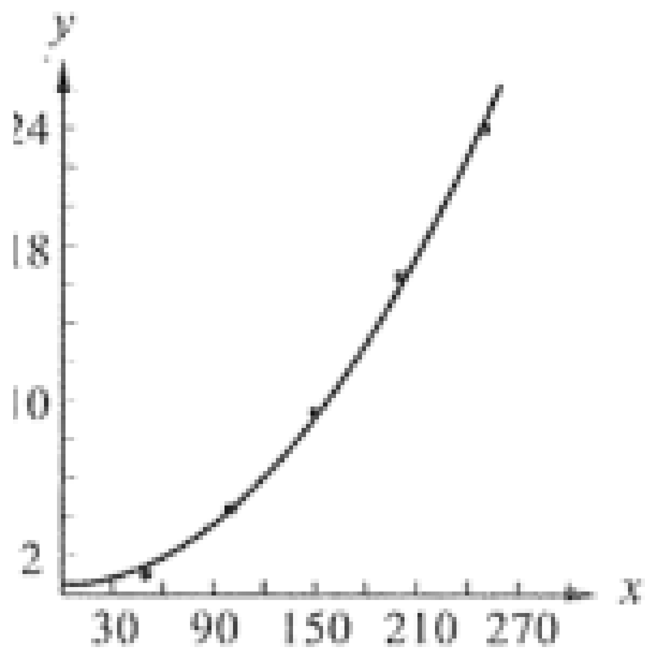
$n = 5$

$$m = \frac{5(2{,}414{,}000) - (137{,}500)(55.2)}{5(61.1875\times10^8) - (137{,}500)^2} = 3.833\times10^{-4}$$

$$b = \frac{(61.1875\times10^8)(55.2) - (2{,}414{,}000)(137{,}500)}{5(61.1875\times10^8) - (137{,}500)^2}$$

$$= 0.50$$

$$y = 0.000383x^2 + 0.50$$



Histogram

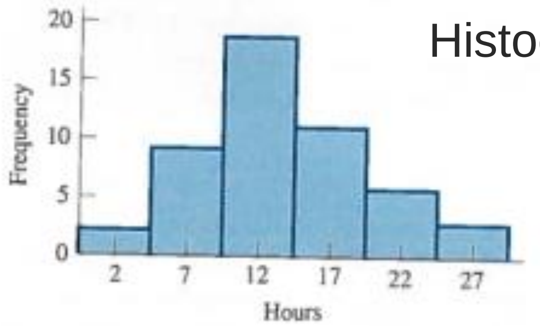


Fig. 22.1

Frequency Polygon

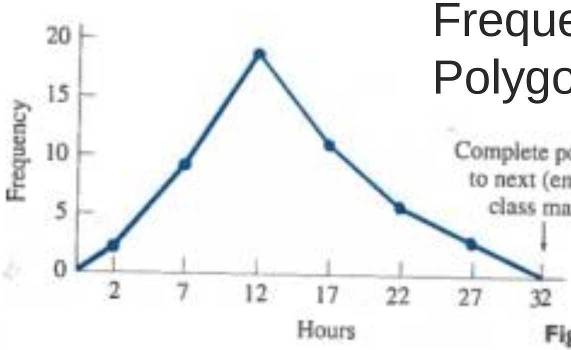


Fig. 22.2

median is the **middle number**

$$\text{Mean} = \frac{\text{Sum of all the values}}{\text{Number of values}}$$

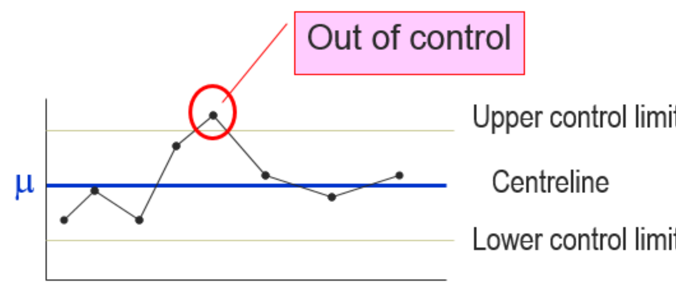
$$\bar{x} = \frac{\Sigma xf}{\Sigma x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$

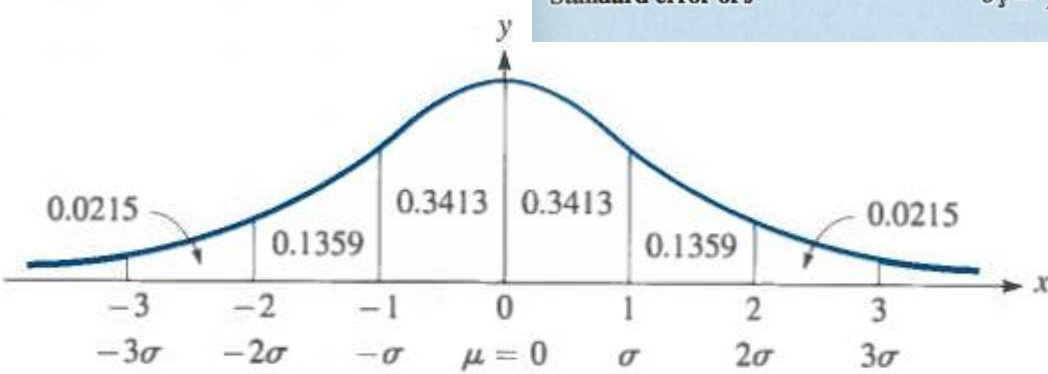
Mode

The value that appears most frequently.

$$s = \sqrt{\frac{\Sigma\left(x - \bar{x}\right)^2}{n-1}}$$

Standard (*z*) score	$z = \dfrac{x - \mu}{\sigma}$
Standard error of $\bar{x}$	$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$
Standard error of *s*	$\sigma_s = \dfrac{\sigma}{\sqrt{2n}}$





$$y = mx + b$$

Arithmetic mean $$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$

Standard deviation $$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

$$s = \sqrt{\frac{n(\sum x^2) - (\sum x)^2}{n(n - 1)}}$$

Normal distribution $$y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

Standard normal distribution $$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Standard (z) score $$z = \frac{x - \mu}{\sigma}$$

Standard error of $\bar{x}$ $$\sigma_x = \frac{\sigma}{\sqrt{n}}$$

Standard error of s $$\sigma_s = \frac{\sigma}{\sqrt{2n}}$$

Least-squares lines $$y = mx + b$$

$$m = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum x^2 - \left(\sum x\right)^2}$$

$$b = \frac{\left(\sum x^2\right)\left(\sum y\right) - \left(\sum xy\right)\left(\sum x\right)}{n\sum x^2 - \left(\sum x\right)^2}$$

Nonlinear curves $$y = m[f(x)] + b$$